Universal Pulse Sequence to Minimize Spin Dephasing in the Central Spin Decoherence Problem

B. Lee,[1,2] W. M. Witzel,[1,3] and S. Das Sarma[1]

[1]*Condensed Matter Theory Center, Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA*
[2]*Science, Mathematics, and Computer Science Magnet Program, Montgomery Blair High School, Silver Spring, Maryland 20901, USA*
[3]*Naval Research Laboratory, Washington, D.C. 20375, USA*

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We present a remarkable finding that a recently discovered [G. S. Uhrig, Phys. Rev. Lett. **98**, 100504 (2007)] series of pulse sequences, designed to optimally restore coherence to a qubit in the spin-boson model of decoherence, is in fact completely model independent and generically valid for arbitrary dephasing Hamiltonians given sufficiently short delay times between pulses. The series maximizes qubit fidelity versus number of applied pulses for sufficiently short delay times because the series, with each additional pulse, cancels successive orders of a time expansion for the fidelity decay. The "magical" universality of this property, which was not appreciated earlier, requires that a linearly growing set of "unknowns" (the delay times) must simultaneously satisfy an exponentially growing set of nonlinear equations that involve arbitrary dephasing Hamiltonian operators.

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Introduction.—The spin-boson (SB) model, where the environment is modeled by a bosonic bath of simple harmonic oscillator quanta, is a famous and extensively used general technique for studying the quantum decoherence of a system coupled to an environment [1]. The model is often used in the context of quantum dissipation and decoherence analyses of wide classes of couplings between a system and its environment. Recently, Uhrig [2] discovered a series of π-pulse sequences that optimally, with respect to the number of pulses, decouple a qubit from a bosonic bath in the SB model, thus protecting the qubit from decoherence. Because errors are inherently introduced by each pulse, the promise of optimal dynamical decoupling (DD) is of great significance to the intense recent activity in quantum information processing where minimizing qubit decoherence and successfully carrying out quantum error correction protocols are crucial.

A particularly important issue in view of the remarkable power of the Uhrig DD (UDD) in fighting qubit decoherence is its applicability in realistic situations beyond the idealized SB model context of its discovery [2]. To understand the general applicability of these sequences, we set out to apply it to a drastically different model of decoherence, the central spin decoherence problem; specifically, we consider the spectral diffusion (SD) process, quantum dephasing of an electron spin qubit coupled to a slowly fluctuating nuclear spin environment (a spin bath). Our investigations led us to discover that UDD transcends all models and is an optimal decoupling sequence for any dephasing Hamiltonian when delay times are sufficiently short because, with each additional pulse, it kills successive orders of a time or Magnus [3,4] expansion [5]. The applicability of the UDD sequence to the SD model, with its extreme contrast to SB, already indicates its powerful generality. What we find is much more general—we show that the Uhrig sequence is model independent.

How to preserve the state of a qubit in a bath is an important theoretical and practical consideration in the field of quantum information. A large energy splitting between the qubit's two logical states (e.g., through the application of a magnetic field for a spin qubit) compared with the temperature of the bath may result in long relaxation (T_1) times; however, the relative phase of a superposition state may not be preserved by this strategy so that dephasing (T_2) decoherence ensues. As a strategy to combat dephasing and an example of DD, a sequence of π pulses may be applied in order to rapidly, on the time scale of the system dynamics, flip the qubit in between time intervals of free evolution (this may be generalized for any quantum system using inverting [6] pulses as a generalization of π pulses). In the simplest case, the Hahn spin echo occurs after applying a single π pulse midway through the system's evolution. Concatenated DD (CDD) [7] sequences can successively improve coherence times, but at the considerable overhead expense of exponentially increasing the number of applied pulses. UDD performs its magic, not only in the SB model but whenever delay times are sufficiently short, with a mere linear scaling in the number of applied pulses.

Pulse sequence echoes.—Considering only dephasing decoherence, the effective Hamiltonian (for any model) may be written in the form $\hat{\mathcal{H}} = \sum_{\pm} |\pm\rangle \hat{\mathcal{H}}_{\pm} \langle\pm|$ with $|+\rangle$ and $|-\rangle$ as the two qubit ket states. For a given pulse sequence with intervals τ_i between successive pulses, the evolution operator is then $\hat{U} = \sum_{\pm} |\pm\rangle \hat{U}_{\pm} \langle\pm|$ (or $\hat{U} = \sum_{\pm} |\mp\rangle \hat{U}_{\pm} \langle\pm|$ if there are an odd number of pulses) with

$$\hat{U}_{\pm} = \ldots \exp(-i\hat{\mathcal{H}}_{\mp}\tau_2) \exp(-i\hat{\mathcal{H}}_{\pm}\tau_1). \quad (1)$$

In order to characterize the coherence decay, we consider the transverse component of the qubit's expectation value; normalized to a maximum of one, the pulse sequence echo v_E is defined in this way such that $v_E = \| \langle \hat{U}_-^\dagger \hat{U}_+ \rangle \| = \| \langle \hat{W} \rangle \|$ where $\hat{W} \equiv \hat{U}_-^\dagger \hat{U}_+$, the $\langle \cdots \rangle$ denotes an appropriately weighted average over the bath states (we use equal weights justified for temperatures large compared to nuclear Zeeman energies), and $\| \cdots \|$ is the magnitude of the resulting complex number.

Uhrig series.—The UDD sequence with n pulses may be defined by [2]

$$\tau_j = \frac{1}{2}\left[\cos\left(\frac{\pi(j-1)}{n+1}\right) - \cos\left(\frac{\pi j}{n+1}\right)\right]t, \qquad (2)$$

for $1 \le j \le n+1$ [corresponding to τ_j in Eq. (1)] where t is the total sequence time. This series was shown [2] to optimally, with respect to number of pulses, suppress decoherence in the SB model. We find, remarkably, that for any form of dephasing Hamiltonian, with no assumptions about $\hat{\mathcal{H}}_\pm$, this sequence yields $v_E = 1 - \mathcal{O}(t^{2n+2})$. Equivalently stated [4], n pulses in the UDD series remove the first n orders of the Magnus expansion of v_E. In comparison with CDD, the number of required pulses scales exponentially with respect to the orders of the Magnus expansion that are cancelled [7]. (It is important to note, however, that CDD can compete with UDD when the Magnus expansion does not converge well.) We will return to a discussion of the universality (Hamiltonian or model independence) of the UDD series at the end of this Letter after we demonstrate its consequence for a model that is drastically different from SB.

Spin boson versus spin bath.—In the SB model, the spin qubit interacts with a bath of noninteracting bosons: $\mathcal{H} = \sum_i \omega_i \hat{b}_i^\dagger \hat{b}_i + \hat{S}_z \sum_i \lambda_i (\hat{b}_i^\dagger + \hat{b}_i)$, where $\hat{b}$ represents boson operators and $\hat{S}_z$ is the z spin operator for the central spin. In contrast, the spin-bath model treats interactions of a central spin, such as a localized electron in a solid, with interacting nuclear spins in a solid-state lattice. Exemplifying the spin-bath model, SD is a dephasing of the central spin as a result of fluctuations of the bath-induced effective magnetic field caused by intrabath interactions. In the limit of a large applied field, the polarization of the central electron spin and the nuclear spins must be individually preserved (the electron having a gyromagnetic ratio that is typically 2000 times larger than those of the nuclei), so that the Hamiltonian is

$$\hat{\mathcal{H}} = \sum_n A_n \hat{S}_z \hat{I}_{nz} + \sum_{n \ne m} (b_{nm} \hat{I}_{n-} \hat{I}_{m+} + c_{nm} \hat{I}_{nz} \hat{I}_{mz}). \qquad (3)$$

Spectral diffusion: Cluster expansion.—Despite the mesoscopic size of the solid-state baths that typically contribute to SD, often involving many millions of nuclear spins, it is feasible to compute $v_E = \| \langle \hat{W} \rangle \|$ with $\hat{W} \equiv \hat{U}_-^\dagger \hat{U}_+$ using a cluster expansion that breaks up the problem into manageable subproblems that each involve only a few nuclei. (This expansion was successfully applied [8] to the problem of Si:P donor electron SD yielding remarkable agreement [9] with experiments [10].) Consider expanding $\hat{W}$ such that $\hat{W} = \sum_{n=0}^{N} \hat{W}^{[n]}$ where $\hat{W}^{[n]}$ contains contributions to $\hat{W}$ that involve n separate clusters of "operatively" interacting nuclei. To be specific, the set of nuclei involved in a term of $\hat{W}^{[1]}$ must all be connected together via factors of bilinear interaction operators to form a single connected cluster. Clusters have spatial proximity when interactions are local. If it is possible to approximate $\langle \hat{W}^{[1]} \rangle$ by only including clusters up to some small size that is much less than the number of nuclei in the bath N, and if the initial bath state is effectively uncorrelated (e.g., a random bath), then $\langle \hat{W}^{[n]} \rangle \approx \langle \hat{W}^{[1]} \rangle^n / n$. In this "cluster approximation," $v_E = \| \langle \hat{W} \rangle \| \approx \exp(\mathrm{Re}\{\langle \hat{W}^{[1]} \rangle\})$. This approximation is extremely useful because it is possible to treat $\langle \hat{W}^{[1]} \rangle$ perturbatively in cases where the perturbation would fail for $\langle \hat{W} \rangle$ directly due to the vast numbers of multiple clusters involved in $\langle \hat{W} \rangle$ (by definition $\hat{W}^{[1]}$ involves only single clusters). In our calculations, we approximate $\langle \hat{W}^{[1]} \rangle$ using a perturbation in cluster size (we include contributions from successively increasing clusters until convergence is achieved). The justification for this perturbation in cluster size is that each additional nucleus in the cluster requires an additional bilinear interaction factor. An expansion in cluster size rather than a direct expansion in orders of intrabath interaction factors (e.g., diagrammatically [11]) is simply more convenient.

Spectral diffusion: Time perturbation.—In addition to the intrabath perturbation, it is also possible for $\langle \hat{W}^{[1]} \rangle$ to converge in a time expansion. The time perturbation applies when the τ_j are small compared to all interactions time scales of the system. The time perturbation is only really relevant if it is applicable on the time scale of the decay (e.g., T_2 observed in spin echo). Different clusters operate, in the sense of contributing to SD decay, on very different time scales depending largely upon differences in the HF interactions (A_n) among cluster nuclei, and these interactions are inhomogeneous over the bath. When there are enough clusters with enough influence operating on the shortest time scales so that these clusters dominate the decay, the decay time may be small compared to all interaction time scales such that the time perturbation is relevant. This, in turn, depends on the distribution of the A_n determined by the shape of the electron wave function. With these considerations and assuming that intrabath interactions are local (e.g., dipolar interactions), electrons in quantum dots with Gaussian-shaped wave functions will tend to, in general, exhibit short-time behavior SD on the decay time scale while donor-bound electrons, with radially exponential wave functions, will not [12]. As an example of a situation in which the time perturbation is *not* appropriate, the Hahn echo decay of donor-bound electrons in Si:P has the form $\exp(-\tau^{2.3})$ [8], which cannot be explained by any Magnus or time expansion. The Hahn

echo decay of GaAs quantum dots, on the other hand, exhibit $\exp(-\tau^4)$ behavior [8] explained by the lowest order of a time perturbation expansion.

Spectral diffusion: Pulse sequences.—Using the cluster expansion technique, we are able to test the effectiveness of DD strategies that use π-pulse sequences in the real-world SD problem. We have previously theoretically verified [13] the powerful effect of SD suppression when applying concatenations of the Hahn echo sequence defined recursively by $p_l := p_{l-1} \to \pi \to p_{l-1} \to \pi$ with $p_0 := \tau$. These CDD sequences cancel successive perturbative orders of both the intrabath [13] and Magnus or time [4,7,14] expansions with each concatenation. The main advantage of this sequence compared with UDD is that it operates on the intrabath perturbation which can be applicable on a much longer time scale, for the time between pulses, than a time expansion (intrabath coupling is on the order of milliseconds while HF interactions limits the time expansion on the order of microseconds). Each concatenation, however, essentially doubles the number of applied pulses, leading to exponential overhead. The main advantage of the UDD series [Eq. (2)] is that it yields successive time expansion cancellations with each added pulse for the SB model, a linear overhead. Figure 1 shows a side-by-side comparison of UDD and CDD sequences.

Spectral diffusion: Results.—We show cluster expansion results (to the lowest nontrivial order in the intrabath perturbation), with coupling constants of Eq. (3) obtained using models described in Ref. [8]. We compare the effects of CDD and UDD sequences on the coherence of both an electron bound to a P donor in natural Si [Fig. 2] and a quantum dot electron in GaAs [Fig. 3], plotted as a function of the total time t of one iteration of the pulse sequence. In Si:P, where only the intrabath perturbation (and not the time perturbation) is applicable, CDD maintains high fidelity (e.g., 10^{-4} decay) for a longer time than UDD. The CDD sequences cancel out perturbative orders of the intrabath coupling with each concatenation [13]; however, UDD sequences cancel only the first and, if there are an even number of pulses, second orders. In the GaAs quantum dot, in which time perturbation is valid, CDD still does better (maintains high fidelity longer) than UDD for equal cancellations of order [Fig. 3(a)]. However, for equal numbers of pulses, UDD preserves coherence far better than CDD [Fig. 3(b)]. In this case, UDD cancels out perturbative orders of a time expansion (exhibited by the successively increasing slopes of the curves) with each extra pulse, exactly what it was shown to do in the SB model. The only significant difference between Si:P and the GaAs quantum dot is the applicability of time perturbation expansion due to their respective electron wave function shapes. Despite the stark, qualitative difference between SB and SD models, UDD proves to be optimal, in the short-time limit (where accessible), for both.

Universality.—What is most striking, however, is that UDD is generically optimal for cancelling orders of a time perturbation for *any* dephasing decoherence. With full generality, let $\hat{\mathcal{H}}_\pm = \hat{X}_0 \pm \hat{X}_1$ without making any assumption about the commutation properties of $\hat{X}_0$ and $\hat{X}_1$. This Hamiltonian, along with the pulse time intervals τ_j for a given pulse sequence will determine the $\hat{U}_\pm$ operators that go into $v_E = \| \langle \hat{W} \rangle \|$ with $\hat{W} \equiv [\hat{U}^-]^\dagger \hat{U}^+$. Taking the real part of $\hat{W}$ yields a more convenient expression, $\mathrm{Re}\{\langle \hat{W} \rangle\} = 1 - \langle \Delta^\dagger \Delta \rangle / 2$ with $\Delta = \hat{U}^+ - \hat{U}^-$.

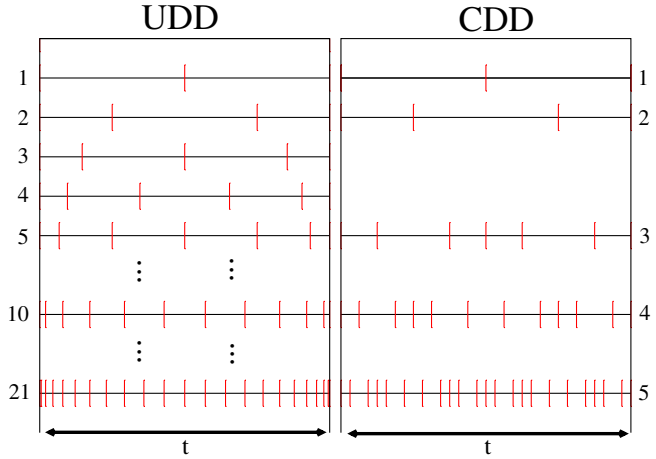

FIG. 1 (color online). Various UDD (left) and CDD (right) pulse sequences in side-by-side comparison where gray (red) lines mark the time of each applied π pulse. The sequences are numbered by their order which corresponds, respectively, to the number of pulses (UDD) or the level of concatenation (CDD). Orders of UDD and CDD with the same number of pulses are lined up horizontally.

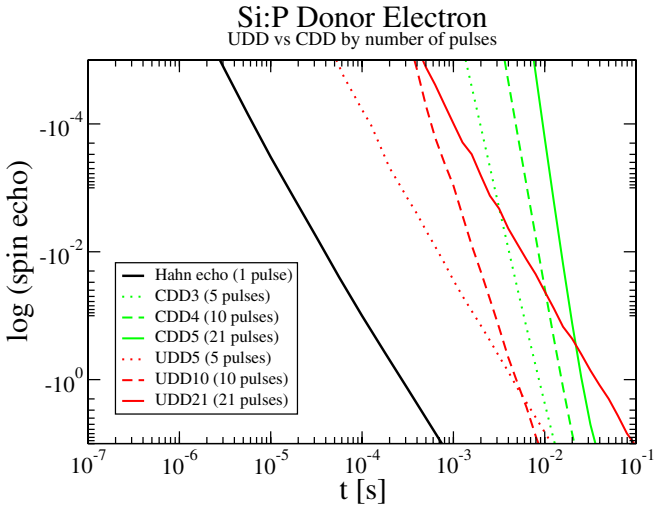

FIG. 2 (color online). Numerical results showing the natural logarithm of echoes from CDD and UDD sequences for Si:P donor electrons in natural Si. Dark gray (red) curves [light gray (green) curves] denote UDD [CDD] sequence results; the Hahn echo decay is shown as a solid black curve for reference. Sequences with the same number of pulses have the same line pattern. UDD has no particular advantage (in the high fidelity regime) to CDD in Si:P, but UDD with an even number of pulses performs better than those with an odd number.

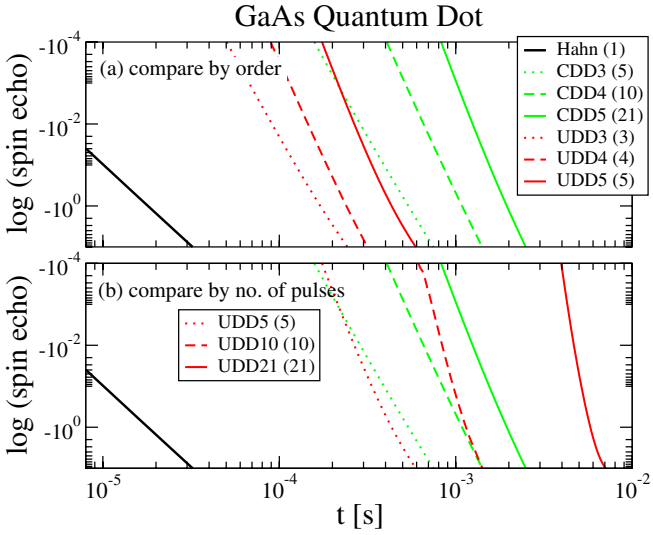

FIG. 3 (color online). Numerical results showing the natural logarithm of echoes from CDD and UDD sequences for quantum dots in GaAs with a 25 nm Fock-Darwin radius, 8.5 nm quantum well thickness. Dark gray (red) curves [light gray (green) curves] denote UDD [CDD] sequence results; the Hahn echo decay is shown as a solid black curve for reference. (a) Sequences of the same order, having the same initial slope, use the same line pattern. (b) Sequences with the same number of pulses have the same line pattern. Note that the Hahn (black) and CDD [light gray (green)] curves in (a) and (b) are the same, shown for comparison with UDD curves.

Since $v_E > \mathrm{Re}\{\langle \hat{W} \rangle\}$, it only serves to make a more pessimistic estimate to ignore the imaginary part of $\langle \hat{W} \rangle$; we may therefore restrict our attention to the real part in our perturbative analysis.

Expanding the exponentials of $\hat{U}_{\pm}$ [Eq. (1)] and collecting terms with common sequences of $\hat{X}_{0,1}$ operators, we define $C_{i_1,i_2,\ldots}$ as the coefficient of a term in $\hat{\Delta} = \hat{U}_+ - \hat{U}_-$ for a corresponding sequence of operators such that

$$\hat{\Delta} = \sum_{m=0}^{\infty} \sum_{i_1,i_2,\ldots,i_m=0}^{1} (-i)^m C_{i_1,i_2,\ldots,i_m} \hat{X}_{i_1} \hat{X}_{i_2} \ldots \hat{X}_{i_m}. \quad (4)$$

These $C\ldots$ coefficients may be written as a sum of products of the τ_j pulse delay times, defined by Eq. (2) for a sequence of n pulses, by expanding Eq. (1). It so happens that for any $m \leq n$, all $C_{i_1,i_2,\ldots,i_m} \sim \mathcal{O}(t^m)$ coefficients are identically zero. We have explicitly proven this for (up to) $n \leq 9$ using computer integer arithmetic by representing each $C\ldots$ as a polynomial of $\alpha = \exp(i\pi/(n+1))$, exploiting the fact that $\alpha^{n+1} = \alpha^{-n-1} = -1$, and noting that two or more points placed symmetrically (i.e., equally spaced) around a circle in the complex plane sum identically to zero. We conjecture that this result is generically true for all integer n, and numerical calculations (up to $n = 14$) are consistent with this assertion. This finding is remarkable. The UDD series satisfies an exponentially growing set of $C\ldots = 0$ nonlinear equations with degrees of freedom (τ_j) that merely grow linearly (one for each pulse).

Conclusion.—We conclude by emphasizing our key finding that the UDD [2] sequences restore coherence optimally and generically in a model-independent manner through successive cancellations in orders of a time or Magnus [4] perturbation expansion with a low overhead of a single pulse for each order of cancellation. While we assume ideal, instantaneous pulses in our analysis, careful pulse shaping can theoretically [15] mitigate effects of the finite width in experimentally applied pulses. The universal Hamiltonian-independent applicability of this series of pulse sequences, originally proposed within the narrow constraint of a spin-boson model, is simply miraculous.

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