Chap 11 - Atom Interferometry

Because of the wave nature of atoms, there can be interesting interference patterns when more than one path is accessible.

Remember $\lambda_{dB} = h/p$ much of the discussion will occur at the semiclassical limit

Much of the discussion is simplified if we understand how a point source of particles behaves at wave number k.

$$\nabla^2 F(\vec{r}) + k^2 F(\vec{r}) = C\,\delta(\vec{r})$$

The $F(\vec{r})$ must be spherically symmetric

In spherical coordinates $\nabla^2 F(\vec{r}) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rF(\vec{r})) +$ angle derivs

$$F(\vec{r}) = \frac{D^+}{r}e^{ikr} + \frac{D^-}{r}e^{-ikr}$$ If you want outgoing waves, $D^- = 0$

How to figure out the relation of D^+ to C?

$$\nabla^2 \frac{1}{r} = ?\,\delta(\vec{r})$$

Set $\frac{1}{r} = \frac{1}{r+\varepsilon}$ and take limit $\varepsilon \to 0^+$

$$S = \frac{1}{r}\frac{\partial^2}{\partial r^2}\left(\frac{r}{r+\varepsilon}\right) = \frac{1}{r}\left(\frac{\partial}{\partial r^2}\left[1 - \frac{\varepsilon}{\varepsilon+r}\right]\right) = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\varepsilon}{(\varepsilon+r)^2}\right) = -\frac{2\varepsilon}{r(\varepsilon+r)^3}$$

$$\iiint S(r)\,d\phi\,d(\cos\theta)\,r^2 dr = -8\pi\varepsilon\int_0^{r_0}\frac{r}{(\varepsilon+r)^3}dr$$

$$= -8\pi\varepsilon\int_0^{r_0}\frac{1}{(\varepsilon+r)^2} - \frac{\varepsilon}{(\varepsilon+r)^3}\,dr$$

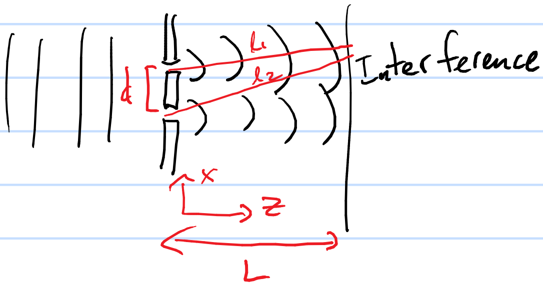
$$= -8\pi\varepsilon\left[-\frac{1}{\varepsilon+r} + \frac{\varepsilon}{2(\varepsilon+r)^2}\right]_0^{r_0} = -4\pi \text{ as } \varepsilon \to 0$$

$$\nabla^2\frac{1}{r} = -4\pi\,\delta(\vec{r})$$ This gives

$$\boxed{F(\vec{r}) = -\frac{C}{4\pi}\frac{e^{ikr}}{r}}$$

[From E&M $\nabla^2 V = -\frac{q\,\delta(\vec{r})}{\varepsilon_0} \Rightarrow V(\vec{r}) = \frac{q}{4\pi\varepsilon_0 r}$]

Two slit interference when size of holes $<< \lambda$. Same probability and phase at each slit.



Amplitude at $z = L$ as a function of x?

$$\text{Amplitude}(x) = \frac{c}{4\pi}\left\{\frac{1}{\ell_1} e^{ik[L^2+(x-\frac{d}{2})^2]^{1/2}} + \frac{1}{\ell_2} e^{ik[L^2+(x+\frac{d}{2})^2]^{1/2}}\right\}$$

If $L >> x$ or d

$$\sqrt{L^2 + (x \pm \tfrac{d}{2})^2} = L\left[1 + \left(\frac{x \pm d/2}{L}\right)^2\right]^{1/2}$$
$$\simeq L + \frac{L}{2}\left(\frac{x \pm d/2}{L}\right)^2 = L + \frac{x^2 + \frac{d^2}{4}}{2L} \pm \frac{xd}{2L}$$

$$\text{Amplitude}(x) = \frac{c}{4\pi \ell} e^{i\phi}\, 2\cos\left(\frac{kxd}{2L}\right) \qquad \phi = k\left[L + \frac{x^2 + d^2/4}{2L}\right]$$

This gives the interference pattern

$$\text{Prob}(x) = |\text{Amp}(x)|^2 \propto \cos^2\left(\frac{kxd}{2L}\right)$$

The peaks are at $kxd/2L = n\pi$

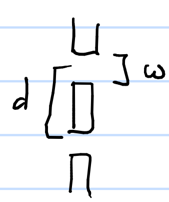
$$\boxed{x = n\frac{2\pi}{k}\frac{L}{d} = n\frac{\lambda L}{d}}$$

See Eq 11.7

Use this formalism to derive the X-ray diffraction from solids. Simplified treatment. pgs 4+5 of PHYS 545 notes

Goes after next 2 pages

How to treat finite slit width



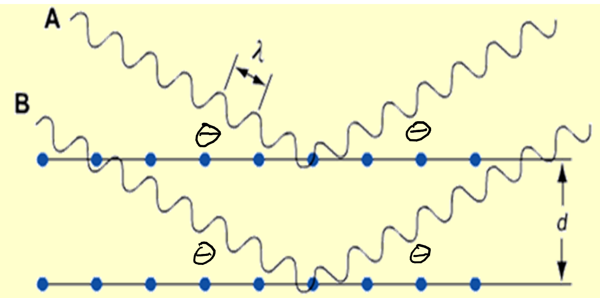
Integrate Amp

$$\text{Amplitude}(x) = \frac{c}{4\pi\ell}\left\{\int_{\frac{d}{2}-\frac{w}{2}}^{\frac{d}{2}+\frac{w}{2}} ds\, e^{ik\left(L + \frac{x^2 + \frac{s^2}{4}}{2L} - \frac{xs}{L}\right)} + \int_{\frac{d}{2}-\frac{w}{2}}^{\frac{d}{2}+\frac{w}{2}} ds\, e^{ik\left(L + \frac{x^2+\frac{s^2}{4}}{2L} + \frac{xs}{L}\right)}\right\}$$

In most situations $y >> s$ which means the s^2 term can be dropped

(This page & next from my PHYS 545 notes)

This distance is directly relevant to X-ray diffraction. Remember Bragg formula



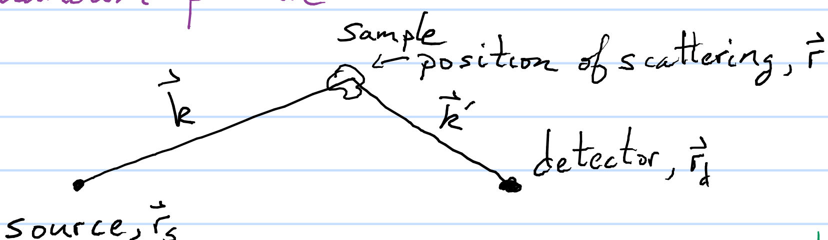
Constructive interference when

$2 d \sin\theta = n\lambda \qquad n = \text{integer}$

While this is correct, can be difficult to use, doesn't give intensity for diff. planes, and difficult to generalize.

We can use quantum mechanics to sketch out a more powerful treatment of X-ray scattering. Assume the scattering amplitude is proportional to the electron density $n(\vec{r})$. Not quite correct because some electrons too deeply bound, depends on polarization of X-rays...

Simplified quantum picture



$\vec{k}$ = initial wave vector of X-ray

$\vec{k}'$ = final wave vector of X-ray

elastic scattering $|\vec{k}| = |\vec{k}'|$

$$\text{Scattering amplitude} = \int \underbrace{A_{\vec{k}}(\vec{r}_s, \vec{r})}_{\text{amp to go from } \vec{r}_s \to \vec{r}} \; \underbrace{A_{\vec{r}}(\vec{k}, \vec{k}')}_{\text{amp to go from } \vec{k} \to \vec{k}'} \; \underbrace{A_{\vec{k}'}(\vec{r}, \vec{r}_d)}_{\text{amp to go from } \vec{r} \to \vec{r}_d} \, d^3\vec{r}$$

Why product of amplitudes?

Why integral?

For the two free space amplitudes $A_{\vec{k}}(\vec{r}_i, \vec{r}_f) = \frac{1}{|\vec{r}_i - \vec{r}_f|} e^{i\vec{k}\cdot(\vec{r}_f - \vec{r}_i)}$

$\vec{k} \parallel \vec{r}_f - \vec{r}_i$?

The amplitude to change directions $A_{\vec{r}}(\vec{k}, \vec{k}') \sim C\, n(\vec{r}) \sim$ electron density

Use the fact that the size of the sample is much smaller than distance between sample/source & sample/detector. This means the $1/|\vec{r}_f - \vec{r}_i|$ are approximately constant.

$$\text{Scattering amplitude} = \frac{1}{|\vec{r}_s - \vec{r}|} \frac{1}{|\vec{r}_d - \vec{r}|} e^{-i\vec{k}\cdot\vec{r}_s} e^{i\vec{k}'\cdot\vec{r}_d}\, C \int e^{i\vec{k}\cdot\vec{r}} e^{-i\vec{k}'\cdot\vec{r}} n(\vec{r})\, d^3\vec{r}$$

Kittel defines the scattering amplitude to be

Scattering amplitude $F = \int dV\, n(\vec r)\, e^{i(\vec k - \vec k')\cdot\vec r}$ where did the other terms go?

$\Delta\vec k = \vec k' - \vec k$ $\qquad F = \int dV\, n(\vec r)\, e^{-i\Delta\vec k\cdot\vec r}$

Here's where the neat trick comes in $n(\vec r) = \sum_G n_{\vec G}\, e^{i\vec G\cdot\vec r}$ $\boxed{\text{periodic!}}$

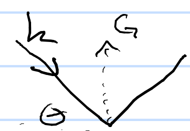
Remember $|\vec k| = \frac{2\pi}{\lambda}$ with $\lambda \sim$ Å and $|\vec G| \sim \frac{2\pi}{\text{lattice spacing}}$

$F = \sum_G \int dV\, n_G\, e^{i(\vec G - \Delta\vec k)\cdot\vec r} \sim 0$ unless $\Delta\vec k = \vec G$ $\quad$ error $\sim \left(\frac{\text{lattice spacing}}{\text{sample size}}\right)^p$

Only get X-ray diffraction when $\vec k' = \vec k + \vec G$ and the amplitude for that scattering is approx proportional to $n_{\vec G}$.

A different looking condition: $\vec k'\cdot\vec k' = k^2 = (\vec k + \vec G)\cdot(\vec k + \vec G) = k^2 + 2\vec k\cdot\vec G + G^2$

For every $\vec G$ there's a $-\vec G \Rightarrow$ $\quad 2\vec k\cdot\vec G = G^2$



Remember $\vec G$ is $\perp$ to lattice plane and has magnitude $\frac{2\pi}{d}n$ ← why factor of n = integer

$2\vec k\cdot\vec G = 2\frac{2\pi}{\lambda}\frac{2\pi n}{d}\sin\theta = \left(\frac{2\pi n}{d}\right)^2 \Longrightarrow 2d\sin\theta = n\lambda$

Another formulation of the diffraction condition starts with $\Delta\vec k = G$

Laue equations $\quad \vec a_1\cdot\Delta\vec k = 2\pi v_1 \quad \vec a_2\cdot\Delta\vec k = 2\pi v_2 \quad \vec a_3\cdot\Delta\vec k = 2\pi v_3$

Another powerful formulation of the diffraction condition from Brillouin

$\vec k\cdot\left(\frac{1}{2}\vec G\right) = \left(\frac{1}{2}\vec G\right)^2 \Rightarrow \vec k = \frac{1}{2}\vec G$ + any amount $\perp$ to $\vec G$

This is how Wigner-Seitz primitive cell was defined (but for lattice)

The systematic construction of these edges (and the volume inside) leads to Brillouin zones. Important for constructing wave functions.

Under this approximation

$$\mathrm{Amp}(x) = \frac{c}{4\pi\ell} e^{i\phi} 2\,\mathrm{Re}\left[\int_{\frac{d}{2}-\frac{w}{2}}^{\frac{d}{2}+\frac{w}{2}} e^{\frac{ikxs}{L}}\, ds\right]$$

$$= \frac{c}{4\pi\ell} e^{i\phi} 2\,\mathrm{Re}\left[e^{ikxd/2L} \frac{L}{iky} e^{\frac{ikxs}{L}}\Big|_{-w/2}^{w/2}\right]$$

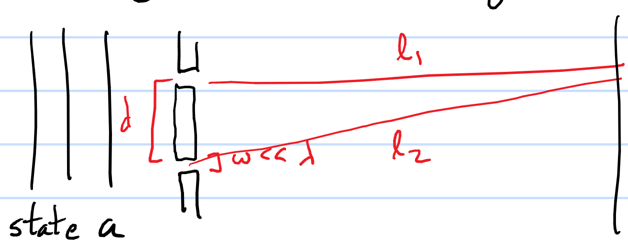
$$= \frac{c}{4\pi\ell} e^{i\phi} 2\,\mathrm{Re}\left[e^{ikxd/2L} \frac{2L}{kx} \sin\left(\frac{kxw}{2L}\right)\right]$$

$$= \frac{c}{4\pi\ell} e^{i\phi} \underbrace{\frac{2L}{kx} \sin\left(\frac{kxw}{2L}\right)}_{\text{single slit diffraction}} \underbrace{\cos\left(kxd/2L\right)}_{\text{2 slit interference}}$$

Diffraction of Buckey balls Why does this work? Buckeyballs have many, many internal states.

Labeling atoms on quantum paths

(slit width small enough to ignore)



Interference

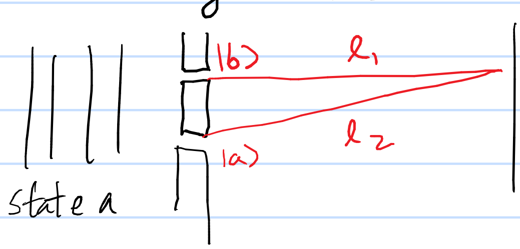
$$\mathrm{Amp}(x) = |a\rangle \frac{c}{4\pi}\left[\frac{1}{\ell_1} e^{ik\ell_1} + \frac{1}{\ell_2} e^{ik\ell_2}\right]$$

Have to average over the coordinates associated with $|a\rangle$ if you don't detect.

$$|\mathrm{Amp}(x)|^2 = \frac{|c|^2}{|4\pi\ell|^2} \cos^2\left(\frac{kxd}{2L}\right) \underbrace{\int \psi_a^*(?)\,\psi_a(?)\,d?}_{=1}$$

= previous result

How to write the amplitude if the internal state is changed to an orthogonal state when going through?



Interference?

$$\mathrm{Amp}(x) = \frac{c}{4\pi}\left[\frac{|b\rangle}{\ell_1} e^{ik\ell_1} + \frac{|a\rangle}{\ell_2} e^{ik\ell_2}\right]$$

How to write the probability? $|a\rangle \Rightarrow \psi_a(s)$ ← internal variable

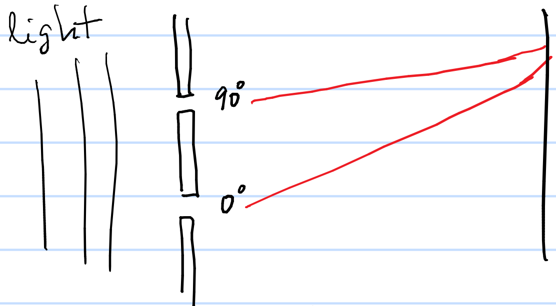
$$\mathrm{Prob}(x) = \left(\frac{c}{4\pi}\right)^2 \int ds \left[\frac{\psi_b^* e^{-ik\ell_1}}{\ell_1} + \frac{\psi_a^* e^{-ik\ell_2}}{\ell_2}\right]\left[\frac{\psi_b e^{ik\ell_1}}{\ell_1} + \frac{\psi_a e^{ik\ell_2}}{\ell_2}\right]$$

The cross terms are 0 after integrating over s

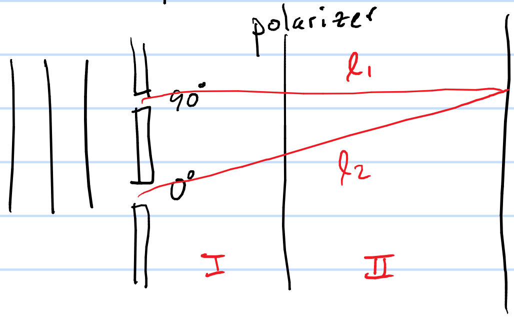
$$\mathrm{Prob}(x) = \left(\frac{c}{4\pi}\right)^2 \left[\frac{1}{\ell_1^2} + \frac{1}{\ell_2^2}\right]$$

No interference! Physical explanation?

Quantum eraser experiment



Interference?



Interference if polarizer at 45°?
" " " " 135°?

erase the slit label

$$|0^\circ\rangle = \frac{1}{\sqrt{2}}|45^\circ\rangle - \frac{1}{\sqrt{2}}|135^\circ\rangle \qquad |90^\circ\rangle = \frac{1}{\sqrt{2}}|45^\circ\rangle + \frac{1}{\sqrt{2}}|135^\circ\rangle$$

$$\mathrm{Amp}_{I}(x) = \frac{c}{4\pi}\left[|90^\circ\rangle \frac{e^{ik\ell_1}}{\ell_1} + |0^\circ\rangle \frac{e^{ik\ell_2}}{\ell_2}\right]$$

$$\mathrm{Amp}_{II,45^\circ}(x) = \frac{c}{4\pi\sqrt{2}}|45^\circ\rangle\left[\frac{e^{ik\ell_1}}{\ell_1} + \frac{e^{ik\ell_2}}{\ell_2}\right]$$

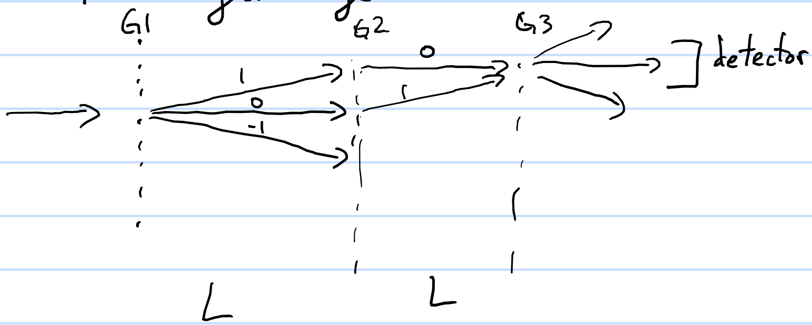
$$\mathrm{Amp}_{II,135^\circ}(x) = \frac{c}{4\pi\sqrt{2}}|135^\circ\rangle\left[\frac{e^{ik\ell_1}}{\ell_1} - \frac{e^{ik\ell_2}}{\ell_2}\right]$$

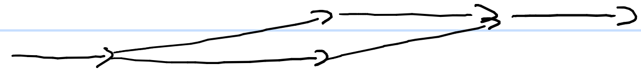
They both give interference, but out of phase

$$\mathrm{Prob}_{II,45^\circ}(x) \propto \cos^2\left(\frac{kxd}{2L}\right)$$
$$\mathrm{Prob}_{II,135^\circ}(x) \propto \sin^2\left(\frac{kxd}{2L}\right)$$

Is there significance that the sum does not have interference?

Three gratings



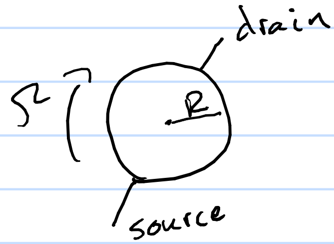


2 paths to same final position $\Rightarrow$ interference

If something is added to one of the paths to change the phase, can relatively easily measure the $\Delta\phi$

This can be a sensitive measure of rotations

nonrotating



rotating

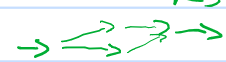
$$\Delta l = \underbrace{\Omega R t}_{\text{clockwise}} - \underbrace{(-\Omega R t)}_{\text{counter clock}}$$

$$t = \frac{\pi R}{v}$$

$$\Delta\phi = k\cdot\Delta l = \frac{2\pi}{\lambda_{dB}}\, 2\,\Omega R\, \frac{\pi R}{v}$$

$$= \frac{4\pi}{\lambda_{dB} v}\,\Omega\,\pi R^2 = \frac{2p}{\hbar v}\,\Omega A = 2\frac{m}{\hbar}\,\Omega A$$

works for



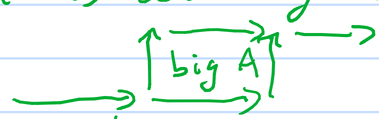
For light $p/v = \frac{pc}{c^2} = \frac{E}{c^2} = \frac{\hbar\omega}{c^2}$

$$\frac{\Delta\phi_m}{\Delta\phi_{light}} = \frac{m}{\hbar\omega/c^2} = \frac{mc^2}{\hbar\omega}$$

$mc^2 \sim 10^{10}$ eV

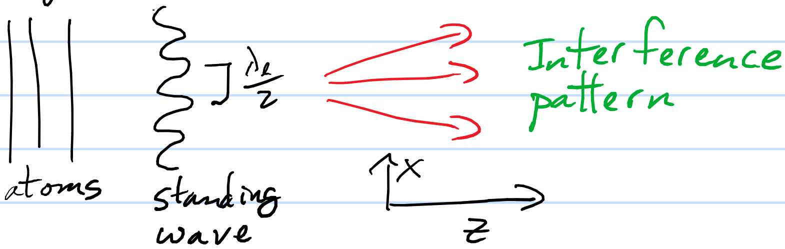
$\hbar\omega \sim 1$ eV

Naively, atoms win by a lot. However, for light can have



Matter + light give comparable sensitivity.

Standing lightwave can be used as a diffraction grating for atoms



First case: no direct absorption of light, $KE_{atom} >> \mathcal{E}_{AC}$
Because the atom energy much larger than the AC Stark shift, the atom crosses the standing wave in a straight line, but acquires a phase (Eikonal approximation)

Before standing wave $\quad \psi(x,z) = e^{ik_z z}$

Just after standing wave $\quad \psi(x,z) \doteq e^{i[k_z z + \Delta\phi(x)]}$

$$\Delta\phi(x) = -\frac{1}{\hbar}\int_{-\infty}^{\infty} \mathcal{E}_{AC}(z(t)) \cos^2\left(\frac{2\pi x}{\lambda_{light}}\right) dt \qquad z(t) = \frac{\hbar k_z}{m} t$$

$$= \phi_0 \cos^2\left(\frac{2\pi x}{\lambda_\ell}\right)$$

Although the atoms moved in a straight line through the standing wave, they acquire momentum in the x direction!

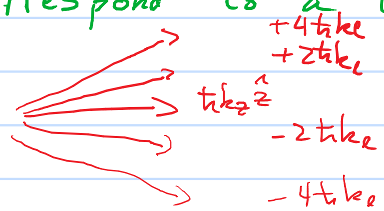
$$\Delta\phi(x) = \phi_0/2 + \frac{\phi_0}{2}\cos\left(\frac{4\pi x}{\lambda_{light}}\right) = \frac{\phi_0}{2} + \frac{\phi_0}{2}\cos(2k_\ell x) \qquad k_\ell = \text{light wave \#}$$

Look at the wave just after the standing wave

$$\psi(x,z) \doteq e^{i\phi_0/2} e^{ik_z z} e^{i\frac{\phi_0}{2}\cos(2k_\ell x)}$$

$$= e^{i\phi_0/2} e^{ik_z z}\left[1 + \frac{i\phi_0}{2}e^{2ik_\ell x} + \frac{i\phi_0}{2}e^{-2ik_\ell x} - \frac{\phi_0^2}{8}\left\{\frac{1}{2} + \frac{1}{2}e^{4ik_\ell x} + \frac{1}{2}e^{-4ik_\ell x}\right\}\ldots\right]$$

Correspond to a wave with momenta $\hbar k_z \hat{z}$, $\hbar k_z \hat{z} \pm 2i\hbar k_\ell \hat{x}$, $\hbar k_z \hat{z} \pm 4i\hbar k_\ell \hat{x}$



Absorbed + emit light

To get to $\hbar k_z \hat{z} + 2\hbar k_\ell \hat{x}$ abs $\uparrow \hat{x}$ $\downarrow -\hat{x}$ emit

Remember standing wave can be thought of as $+\hat{x}$ and $-\hat{x}$ lasers

Because $k_\ell \ll k_z$, this gives a series of "interference" peaks at

$$\sin\theta_n = n\frac{2k_\ell}{k_z} \quad \Rightarrow \quad \frac{\lambda_\ell}{2}\sin\theta_n = n\,\lambda_{dB}$$

Case 2: The light "destroys" the atom. For example, light leads to ionization. Absorbtion rate $\Gamma(I) = \Gamma_0 I/I_0$

$$\psi = |a\rangle e^{ik_z z} \quad \text{before the standing wave}$$

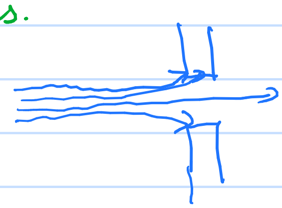
$$\psi = |a\rangle e^{-(\Gamma_0 I(x)/I_0)t/2} e^{ik_z z} \quad \text{just after the standing wave}$$

$$\frac{\Gamma_0 I(x)}{I_0}\frac{t}{2} = \alpha \cos^2\left(\frac{2\pi x}{\lambda_\ell}\right)$$

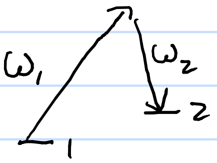
Perform the same type of analysis as case 1. Gives diffraction same diffraction peaks but with different strengths (absorption in case 2 vs. phase shift in Case 1)

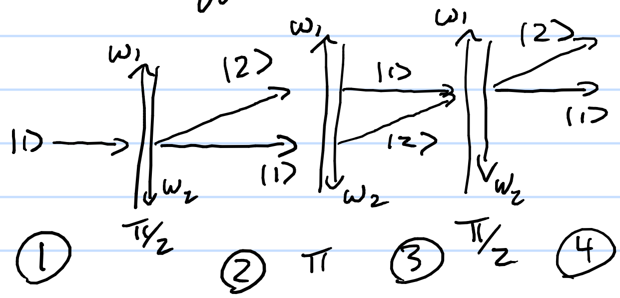
Diffracting from light vs through a material grating

Material grating wins on spacing is smaller $d < \lambda_\ell/2$ by a factor of few.
Material grating loses on atoms sticking to walls, also how to deal with large molecules (size through hole, more attracted to walls.



Can use the Raman effect to make interferometer.





① $\psi = |1\rangle e^{ik_z z}$

② $\psi = \frac{1}{\sqrt{2}}|1\rangle e^{ik_z z} + \frac{e^{i\phi_{12}}}{\sqrt{2}}|2\rangle e^{i(k_z z + 2k_\ell x)}$ $\quad\phi_{12}$ from lasers

③ $\psi = \frac{1}{\sqrt{2}}|2\rangle e^{i(k_z z + 2k_\ell x)} + \frac{e^{i(\phi_{12}+\phi_{23})}}{\sqrt{2}}|1\rangle e^{ik_z z}$ $\quad\phi_{23}$ from lasers + path

④ $\psi = \frac{e^{i\phi_p}}{2}\left[|1\rangle e^{i\phi_{34}} e^{ik_z z} + |2\rangle e^{i(k_z z + 2k_\ell x)}\right]$

$+ \frac{1}{2} e^{i(\phi_{12}+\phi_{23})}\left[|1\rangle e^{ik_z z} - |2\rangle e^{-i\phi_{34}} e^{i(k_z z + 2k_\ell x)}\right]$ $\quad\phi_{34}$ from laser, ϕ_p from path

$= \frac{1}{2}\left(e^{i(\phi_p+\phi_{34})} + e^{i(\phi_{12}+\phi_{23})}\right)|1\rangle e^{ik_z z}$

$+ \frac{1}{2}\left(e^{i\phi_p} - e^{i(\phi_{12}+\phi_{23}-\phi_{34})}\right)|2\rangle e^{i(k_z z + 2k_\ell x)}$

The probability to finish in state 1

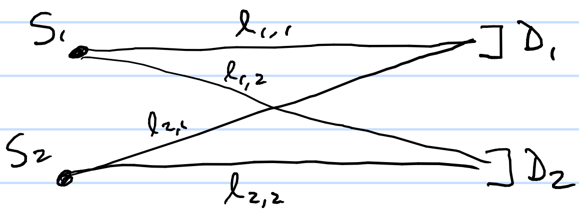
$$P_1 = \cos^2\left(\frac{\phi_{12}+\phi_{23}-\phi_{34}-\phi_p}{2}\right)$$

The probability to finish in state 2

$$P_2 = \sin^2\left(\frac{\phi_{12}+\phi_{23}-\phi_{34}-\phi_p}{2}\right)$$

Very easy to detect whether in state 1 or 2.
Have a laser drive a transition $|1\rangle \to |\text{excited}\rangle$ or $|2\rangle \to |\text{excited}\rangle$ and detect scattered photons

The Hanbury-Brown and Twiss effect is a counter intuitive effect that is similar in spirit to quantum optics effects like photon antibunching. Will first look at the classical derivation. The basic idea is that light from two unrelated sources can have <u>intensity</u> correlation at separated detectors. Caused much controversy before being understood.



S_i = source i
D_i = detector i
l_{ij} = distance from source i to detector j

Both sources emit waves. For simplicity take their intensity to be the same. The frequencies must be the same. <u>BUT</u> they can have a random phase relative to each other: ϕ

At detector i, the amplitude is

$$\mathrm{Amp}(i) = \frac{C}{l_{1,i}} e^{i(kl_{1,i} + \phi - \omega t)} + \frac{C}{l_{2,i}} e^{i(kl_{2,i} - \omega t)}$$

The phase ϕ varies randomly

The E-field at detector i is obtained from the real part. Also use $l_{i,j} = l$ when not in the exponent.

$$E(i) = \frac{C}{l}\left[\cos(kl_{1i} + \phi - \omega t) + \cos(kl_{2i} - \omega t)\right]$$

The intensity at i is the cycle average of $E^2(i)$

$$I(i) = \tilde{C}\frac{\omega}{2\pi}\int_0^{2\pi/\omega} \left[\cos^2(kl_{1i} + \phi - \omega t) + \cos^2(kl_{2i} - \omega t) + 2\cos(kl_{1i} + \phi - \omega t) \times \cos(kl_{2i} - \omega t)\right] dt$$

$$= \tilde{C}\left[\frac{1}{2} + \frac{1}{2} + \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \cos(kl_{1i} + kl_{2i} + \phi - 2\omega t) + \cos(kl_{1i} - kl_{2i} + \phi)\, dt\right]$$

$$= \tilde{C}\left[1 + 0 + \cos(kl_{1i} - kl_{2i} + \phi)\right]$$

If you average over ϕ, the intensity is the same in both detectors. No interference as expected. However...

look at

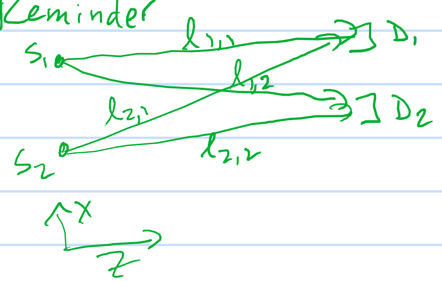
$$\langle (I(1) - \langle I(1)\rangle)(I(2) - \langle I(2)\rangle)\rangle = \tilde{C}^2 \cos(k(\ell_{11} - \ell_{21}) + \phi)\cos(k(\ell_{12} - \ell_{22}) + \phi)$$

$$= \frac{1}{2}\tilde{C}^2 \left\langle \cos\left[k(\ell_{11} + \ell_{12} - \ell_{21} - \ell_{22}) + 2\phi\right] + \cos\left[k(\ell_{11} - \ell_{12} - \ell_{21} + \ell_{22})\right]\right\rangle$$

When you average over the phase difference, the last term survives

$$\langle (I(1) - \langle I(1)\rangle)(I(2) - \langle I(2)\rangle)\rangle = \frac{\tilde{C}^2}{2}\cos\left[k(\ell_{11} - \ell_{12} - \ell_{21} + \ell_{22})\right]$$

Reminder



Let the source be in the x,y plane but the detectors are in x

$$\ell_{i,j} = \sqrt{(x_i' - x_j)^2 + y_i'^2 + L^2}$$

$$\cong L + \frac{1}{2}\frac{(x_i' - x_j)^2 + y_i'^2}{L}$$

$$\ell_{1,1} - \ell_{1,2} - \ell_{2,1} + \ell_{22} = \frac{1}{2L}\left[(x_1' - x_1)^2 + y_1'^2 - (x_1' - x_2)^2 - y_1'^2 - (x_2' - x_1)^2 - y_2'^2 + (x_2' - x_2)^2 + y_2'^2\right]$$

$$= \frac{1}{2L}\left[-2x_1'x_1 + 2x_1'x_2 + 2x_2'x_1 - 2x_2'x_2\right]$$

$$= -\frac{1}{L}(x_2 - x_1)(x_2' - x_1')$$

note the y-dependence dropped out

Put together with the earlier relation

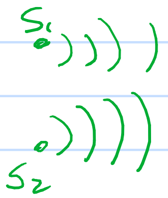
$$\langle \Delta I_1 \Delta I_2\rangle = \frac{\tilde{C}^2}{2}\cos\left(\frac{2\pi}{\lambda}\frac{(x_2 - x_1)(x_2' - x_1')}{L}\right)$$

The interference depends on the angular separation of the source points $\Delta\theta = (x_2' - x_1')/L$

The Hanbury-Brown-Twiss effect has been used to estimate the size of many things (for example, nuclei)

Now turn to the quantum version. Can do the case for identical Bosons or Fermions in the same derivation.

Look at the 2 particle wave function when 1 particle is emitted from source 1 and one from source 2



$$\psi_{b/f}(\vec{r}_1, \vec{r}_2) = C \left[\frac{e^{ik|\vec{r}_1 - \vec{r}_1'|}}{|\vec{r}_1 - \vec{r}_1'|} \frac{e^{ik|\vec{r}_2 - \vec{r}_2'|}}{|\vec{r}_2 - \vec{r}_2'|} \pm \frac{e^{ik|\vec{r}_2 - \vec{r}_1'|}}{|\vec{r}_2 - \vec{r}_1'|} \frac{e^{ik|\vec{r}_1 - \vec{r}_2'|}}{|\vec{r}_1 - \vec{r}_2'|} \right]$$

particle 1 from S_1
" 2 " S_2

why?

particle 2 from S_1
" 1 " S_2

Now look at the probability that particle 1 goes into detector 1 and particle 2 goes into detector 2. Use the notation
$P(i,j)$ is the probability particle 1 goes into detector i and particle 2 goes into detector j

some const.

$$P(1,2) = D \left| e^{ikl_{1,1}} e^{ikl_{2,2}} \pm e^{ikl_{1,2}} e^{ikl_{2,1}} \right|^2$$
$$= D \left\{ 1 + 1 \pm 2\cos\left[k\left(l_{1,1} + l_{2,2} - l_{1,2} - l_{2,1}\right)\right] \right\}$$
$$= 2D \left[1 \pm \cos \Delta\phi_{HBT} \right]$$

$\Delta\phi_{HBT}$ = the Hanbury-Brown-Twiss phase

$$P(2,1) = P(1,2)$$

What's the probability that 1 particle goes into each detector?

$$P_{1\,each} = 4D \left[1 \pm \cos \Delta\phi_{HBT} \right]$$

Now compute the probability 2 goes into 1 detector.

$$P(1,1) = D \left| e^{ikl_{1,1}} e^{ikl_{2,1}} \pm e^{ikl_{1,1}} e^{ikl_{2,1}} \right|^2 = D\,|1 \pm 1|^2 = 2D(1 \pm 1)$$

$$P(2,2) = P(1,1)$$

Note the interesting effect: the probability for 2 Fermions going into the same "point" detector is 0. Actually, nonzero for finite size detector. Integrate $1 - \cos\Delta\phi_{HBT}$ over detector size

Now look at the HBT parameter $\langle I_1 I_2 \rangle$

$$\langle I_1 I_2 \rangle = 1 \cdot 1 \, (P(1,2) + P(2,1)) + 2 \cdot 0 \, (P(1,1) + P(2,2))$$
$$= 4D \, [1 \pm \cos \Delta\phi_{HBT}]$$

This is strange. The quantum Boson case gives same expression as the classical expression for light.

Actually this is the $\langle (I_1 - \langle I_1 \rangle)(I_2 - \langle I_2 \rangle) \rangle$ classical result because I dropped the case where 1 particle goes into the detector and the other has nothing. My quantum $\langle I_1 I_2 \rangle$ is actually $\langle (I_1 - \langle I_1 \rangle)(I_2 - \langle I_2 \rangle) \rangle$

The fact that the intensity correlation gave an interesting result inspired theorists (including Glauber) to develop quantum theory of "atomic" QED: photon correlations, coherent states, etc.

Mention photon antibunching