PHYS515 (Spring 2019), Test #1, V1 Name: ____________

Show work to get credit. In probs 1, 4, 6, your answer should have number and units.

Prob 1.50 (1) (5 pts) 2 moles of CO_2 goes to ??? moles of CO and ??? moles of O_2 at constant pressure. (a) How many moles of CO and O_2 are produced? (b) What is the change in internal energy?

$$2\,CO_2 \rightarrow n_{co}\,CO + n_{O_2}\,O_2$$

(a) C: $2 = n_{co} \Rightarrow \underline{n_{co} = 2}$ O: $4 = n_{co} + 2n_{O_2} \Rightarrow \underline{n_{O_2} = 1}$

Constant pressure $\Rightarrow$ use enthalpy

$$\Delta U + P\Delta V = 2\cdot(-110.53\text{ kJ}) + 1\cdot 0\text{ kJ} - 2(-393.51)\text{ kJ}$$

$$= 565.96\text{ kJ}$$

$$-P\Delta V = PV_i - PV_f = n_i RT - n_f RT = (n_i - n_f)RT$$

$$= (2\text{ mol} - 3\text{ mol})\, 8.31\frac{J}{K}\, 298K = -2.48\text{ kJ}$$

$$\Delta U = (565.96 - 2.48)\text{ kJ} = \underline{563.48\text{ kJ}}$$

V2 $\Delta U = \underline{-563.48\text{ kJ}}$ (every step opposite)

Class (2) (5 pts) You measure the current, I, through an inductor with inductance L. It has probability distribution $\mathcal{P}(I) = Ae^{-LI^2/(2k_BT)}$. (a) Determine A. (b) Determine the average of I and the average of I^2.

The A is determined by normalization

a) $$\int_{-\infty}^{\infty} P(I)\,dI = 1 = A\int_{-\infty}^{\infty} e^{-\frac{LI^2}{2k_BT}}\,dI = A\sqrt{\frac{2k_BT\pi}{L}} \Rightarrow A = \sqrt{\frac{L}{2k_BT\pi}}$$

b) $$\overline{I} = A\int_{-\infty}^{\infty} I\, e^{-\frac{LI^2}{2k_BT}}\,dI = \underline{0}$$

$$\overline{I^2} = A\int_{-\infty}^{\infty} I^2 e^{-\frac{LI^2}{2k_BT}}\,dI = A\,\frac{1}{2}\sqrt{\frac{2\pi k_BT}{L}}\,\frac{2k_BT}{L} = \underline{\frac{k_BT}{L}}$$

V2 $A = \underline{\frac{1}{\sqrt{2\pi k_BTC}}}$ $\quad \underline{\overline{q} = 0}$ $\quad \underline{\overline{q^2} = Ck_BT}$

Prob 2.26 (3) (5 pts) In atomic experiments, it is possible to confine the atoms so that they can only move in 1D (the other two dimensions are frozen out). Give the entropy for neon atoms confined to 1D. There are N atoms confined to a length, L, with internal energy U where k_BT is much larger than the spacing of quantum levels.

Repeat the derivation of Sec 2.5 except $V \to L$ and $d = N$

$$\Omega_N \simeq \frac{1}{N!}\frac{L^N}{h^N}\frac{\pi^{N/2}}{(N/2)!}\left(\sqrt{2mU}\right)^N = \left(\frac{\sqrt{2mU\pi}\,L}{h}\right)^N \frac{1}{\left(\frac{N}{e}\right)^N\sqrt{2\pi N}\,\left(\frac{N}{2e}\right)^{N/2}\sqrt{2\pi N/2}}$$

$$= \left(\frac{\sqrt{4m\pi U}\;L e^{3/2}}{h\;N^{3/2}}\right)^N \frac{\sqrt{2}}{2\pi N} \to \text{can drop}$$

$$S = k_B \ln\Omega = N k_B \ln\left[\frac{\sqrt{2m\pi U/N}}{h}\frac{L}{N}e^{3/2}\right]$$

$$= N k_B\left\{\ln\left[\frac{\sqrt{2m\pi U/N}}{h}\frac{L}{N}\right] + \frac{3}{2}\right\}$$

Prob 3.1 (4) (5 pts) System a has 2 harmonic oscillators and system b has 3 harmonic oscillators. There are a total of 7 excitations. (a) Determine the most likely and the least likely macrostates. (b) Determine the probability to be in each of these states.

a) $\Omega_a = \frac{(q_a+1)!}{q_a!\,1!} = q_a + 1 \qquad \Omega_b = \frac{(q_b+2)!}{q_b!\,2!} = \frac{1}{2}(q_b+2)(q_b+1)$

Total multiplicity of macrostate is $\Omega = \Omega_a\Omega_b$

Least must be $q_a = 0,\ q_b = 7$ $(\Omega = 1\cdot\frac{1}{2}9\cdot 8 = 36)$ or $q_a = 7,\ q_b = 0$ $(\Omega = 8\cdot 1 = 8)$

$q_a = 7,\ q_b = 0$ has smaller multiplicity

To find max test others $q_a = 1, q_b = 6$ $\ \Omega = 2\cdot\frac{1}{2}8\cdot 7 = 56$;

$q_a = 2,\ q_b = 5$ $\quad \Omega = 3\cdot\frac{1}{2}7\cdot 6 = 63 \leftarrow$ max

$q_a = 3,\ q_b = 4$ $\quad \Omega = 4\cdot\frac{1}{2}6\cdot 5 = 60$

b) The total number of states $\Omega_{TOT} = \frac{(q+4)!}{q!\,4!} = \frac{(q+4)(q+3)(q+2)(q+1)}{4\cdot 3\cdot 2\cdot 1}$

$$= \frac{11\cdot 10\cdot 9\cdot 8}{4\cdot 3\cdot 2\cdot 1} = 330$$

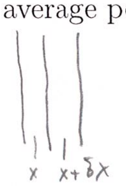
$P_{small} = \frac{8}{330} = 0.024 \qquad P_{large} = \frac{63}{330} = 0.191$

Version 2 $\quad q_a \longleftrightarrow q_b$

Prob 1.16

(5) (10 pts) You have an ideal gas in thermal equilibrium at a temperature T. Each atom is confined to a box in y, z with area A and experiences an external force $F(x) = -\partial PE(x)/\partial x$ which depends on x where $PE(x)$ is the potential energy. The density and pressure depend on x but not on y, z, t. (a) Determine the differential equation for the pressure as a function of x. (Hint: the force on the gas from the changing pressure must balance the external force or the gas will flow.) (b) Solve the differential equation for the pressure as a function of x if the pressure at $x = 0$ is $P(0)$. (c) Give the number density as a function of x in terms of the number density at $x = 0$. (d) For a harmonic potential, $PE(x) = (1/2)M\omega^2 x^2$, determine the number density versus x if there are N atoms. (e) From the result of (d), determine the average potential energy of a single atom.

(a)



Force on midplane $= A \cdot P(x) - A\,P(x+\delta x)$

$$= -A \frac{dP}{dx} dx$$

This must be balanced by external force

$$-A \frac{dP}{dx} \delta x + F(x) \underbrace{\frac{N}{V} A \delta x}_{\text{number of atoms}} = 0$$

Use $\frac{N}{V} = \frac{P(x)}{k_B T} \Rightarrow \underline{\frac{dP}{dx} = \frac{P(x)}{k_B T} F(x) = -\frac{P(x)}{k_B T} \frac{dPE(x)}{dx}}$

(b) $$\frac{dP}{P} = -\frac{1}{k_B T} \frac{dPE(x)}{dx} dx \Rightarrow \ln\left(\frac{P(x)}{P(0)}\right) = -\frac{1}{k_B T}\left[PE(x) - PE(0)\right]$$

$$\underline{P(x) = P(0)\, e^{-\frac{PE(x) - PE(0)}{k_B T}}}$$

(c) $$\frac{N(x)}{V} = \frac{P(x)}{k_B T} = \frac{P(0)}{k_B T} e^{-\frac{PE(x) - PE(0)}{k_B T}} = \underline{\frac{N(0)}{V} e^{-\frac{PE(x) - PE(0)}{k_B T}}}$$

(d) $$\int_{\text{all space}} \frac{N(x)}{V} dx\,dy\,dz = N = \frac{N(0)}{V} A \int_{-\infty}^{\infty} e^{-\frac{m\omega^2}{2k_B T} x^2} dx = \frac{N(0)}{V} A \sqrt{\frac{2k_B T \pi}{m\omega^2}}$$

$$\Rightarrow \frac{N(0)}{V} = \frac{N}{A}\sqrt{\frac{m\omega^2}{2k_B T \pi}} \qquad \left(\frac{N(x)}{V}\right) = \frac{N}{A}\sqrt{\frac{m\omega^2}{2k_B T \pi}}\, e^{-\frac{\frac{1}{2}m\omega^2 x^2}{k_B T}}$$

(e) $$\overline{PE} = \frac{1}{N}\int \frac{N(x)}{V} PE(x)\, dx\,dy\,dz = \sqrt{\frac{m\omega^2}{2k_B T \pi}}\, k_B T \int_{-\infty}^{\infty} \frac{\frac{1}{2}m\omega^2 x^2}{k_B T} e^{-\frac{\frac{1}{2}m\omega^2 x^2}{k_B T}} dx$$

$$= k_B T \frac{1}{\sqrt{\pi}} \frac{1}{2}\sqrt{\pi} = \underline{\frac{k_B T}{2}}$$

Equipartition!

class (6) (10 pts) When gas 1 has volume V_1, N_1 atoms, and total internal energy U_1, the multiplicity is $f_1(N_1)V_1^{N_1}U_1^{3N_1/2}$. When gas 2 has volume V_2, N_2 molecules, and total energy U_2, the multiplicity is $f_2(N_2)V_2^{N_2}U_2^{7N_2/2}$. Gas 1 and 2 are in thermal equilibrium, the total volume of this system is V, and the total internal energy of this system is U. (a) What is the average value of U_1 and what is its spread for *any* large N_1 and N_2? (Hint: the average value of $U_1 \neq U/2$.) (b) If $N_1 = N_2$, $V = 8\ \text{m}^3$, P is 1.5 atm., and T is room temperature, what is the relative uncertainty in U_1? This should be an actual number (for example 1.8×10^{-3}).

The total multiplicity $\Omega = \Omega_1 \Omega_2 = f_1(N_1) f_2(N_2) V_1^{N_1} V_2^{N_2} U_1^{\frac{3N_1}{2}} (U-U_1)^{\frac{7N_2}{2}}$

The average value of U_1 is when $\frac{\partial \Omega}{\partial U_1} = 0$

$$\frac{\partial \Omega}{\partial U_1} = \frac{3N_1}{2U_1}\Omega - \frac{7N_2}{2(U-U_1)}\Omega = 0 \Rightarrow \frac{7N_2}{2}U_1 = \frac{3N_1}{2}(U-U_1)$$

$$\bar{U}_1 = \frac{3N_1}{7N_2+3N_1} U \qquad \left(\text{In version 2} \quad \bar{U}_1 = \frac{3N_1}{5N_2+3N_1} U\right)$$

We know $\Omega \simeq \Omega(\bar{U}_1)\, e^{-(U_1-\bar{U}_1)^2/\Delta U^2} \Rightarrow \left.\frac{d^2\Omega}{dU_1^2}\right|_{U_1=\bar{U}_1} = -\frac{2}{\Delta U^2}\Omega(\bar{U}_1)$

$$\Delta U^2 = -\frac{2\Omega(\bar{U}_1)}{\frac{d^2\Omega}{dU_1^2}}$$

$$\frac{d^2\Omega}{dU_1^2} = -\frac{3N_1}{2U_1^2}\Omega - \frac{7N_2}{2(U-U_1)^2}\Omega + \cancel{\left(\frac{3N_1}{2U_1} - \frac{7N_2}{2(U-U_1)}\right)\Omega'} \to 0$$

$$= -\left[\frac{(7N_2+3N_1)^2}{U^2 6N_1} + \frac{(7N_2+3N_1)^2}{U^2 14N_2}\right]\Omega(\bar{U}_1) = -\frac{1}{2}\frac{(7N_2+3N_1)^3}{U^2 21 N_1 N_2}\Omega(\bar{U}_1)$$

$$\Delta U^2 = U^2 \frac{42 N_1 N_2}{(7N_2+3N_1)^3} \qquad \text{in version 2} \quad \Delta U^2 = U^2 \frac{30 N_1 N_2}{(5N_2+3N_1)^3}$$

(b) $P_1V_1 = N_1 k_B T \Rightarrow N_1 = \frac{P_1V_1}{k_B T} = \frac{1.5\times10^5 \cdot 4}{1.38\times10^{-23}\cdot 300} = 1.45\times10^{26}$

V2 $\to N_1 = 8.7\times10^{25}$

$$\frac{\Delta U}{U} = \left(\frac{0.042}{1.45\times10^{26}}\right)^{1/2} = 1.7\times10^{-14}$$

(in version 2 $\quad \frac{\Delta U}{U} = \left(\frac{30}{8^3\cdot 8.7\times10^{25}}\right)^{1/2} = 2.6\times10^{-14}$)