PHYS 416 (Fall 2025) Test 1 Name: ____________________

Show work to receive full credit. Problems 1,2,4 require numerical answers.

5 pts 1. You have an object at 25 °C and one atmosphere pressure. You manipulate the object so the volume changes from 2.00010 m^3 to 2.00000 m^3 with no change to the internal energy or number of particles. a) What is the change in the entropy of the object? b) What is the change in multiplicity of the object?

5 pts 2. You have 5.0×10^{24} He atoms at 25 °C and one atmosphere pressure. a) What is the volume of the container? b) You add 60 J energy to the He leaving the volume and number of atoms unchanged. What is the change in pressure?

5 pts 3. A Robon is a quantum system with 3 states with energies $E_0 = 0$, $E_1 = \varepsilon$, and $E_2 = 2\varepsilon$. A system consists of 3 Robons with each of the possible states equally likely. a) What is the total number of available states? b) Make a table that shows the probability for the system to have a total energy of 0ε, 1ε, ..., 6ε.

5 pts 4. You have a gas of He that has been enhanced so that 100% is ^{3}He. ^{3}He atoms have spin 1/2 and ^{4}He atoms have spin 0. There's no energy associated with the spin. Suppose you have one mole of this mixture in a container of volume, V, at 300 K and atmospheric pressure. How much more/less (or the same) entropy does this mixture have compared to 100% ^{4}He at the same temperature and volume?

10 pts 5. System **a** consists of N_a two state systems with $E_1 = 0$ and $E_2 = \varepsilon$; the number in state 2 is N_2. System **b** consists of N_b quantum harmonic oscillators with total number of excitations q_b and energy $q_b hf$. For this problem, $\varepsilon = \phi hf$ and both N_a and N_b are large numbers and of comparable values (say $0.5N_b < N_a < 2N_b$). The two systems are in thermal contact. a) For a total energy $E = Qhf$, give a single equation for q_b in terms of N_a, N_b, ϕ, Q. b) For $\phi = 1$, find q_b in terms of N_a, N_b, Q.

10 pts 6. An "atom" has only two energies $E_1 = 0$ and $E_2 = \varepsilon$. There are M_1 states with energy E_1 and M_2 states with energy E_2. (Each "atom" has $M_1 + M_2$ states.) The system is composed of N of these "atoms" and the total energy is $E = N_2\varepsilon$ with N_2 the number of "atoms" with energy E_2. N and E/ε are large numbers. a) Determine the entropy in terms of E, N, ... b) Determine the temperature in terms of these parameters. c) What is the probability that an "atom" has energy E_2? **The test had E_2 but it should have been ε. I did not grade this part.** d) Determine the change in entropy when one "atom" is added to the system keeping the total energy fixed.

Prob 1 The idea behind this is Eq. (3.57)

a) $\Delta S = \frac{1}{T}\cancel{\Delta U}^{\to 0} + \frac{P}{T}\Delta V - \frac{\mu}{T}\cancel{\Delta N}^{\to 0} = \frac{1.01\times10^{5}\,Pa}{298\,K}(-10^{-4}) = -0.034\ J/K$

b) $S = k\ln\Omega$ Eq (2.45) $\Rightarrow$ $\Omega_f = \Omega_i\, e^{\Delta S/k} = \Omega_i\, e^{(-0.034/1.38\times10^{-23})} = \Omega_i e^{-2.5\times10^{21}}$

Prob 2 Use ideal gas law Eq (1.5) $PV = NkT$

a) $V = \frac{5\times10^{24}\ 1.38\times10^{-23}\,J/K\ 298\,K}{1.01\times10^{5}\,Pa} = 0.204\ m^3$

b) $(P+\Delta P) = \frac{Nk(T+\Delta T)}{V} \Rightarrow \Delta P = \frac{Nk\Delta T}{V}$

Find the ΔT from the equipartition theorem, Eq 1.23 with $f=3$

$Nk\Delta T = \frac{2}{3}\Delta U = \frac{2}{3}\,60\,J = 40\,J$

$\Delta P = 40\,J / 0.204\ m^3 = 196\ Pa$

Prob 3 a) The total number of states is $3^3 = 27$. (number of states for 1 Robon; # Robons)

b) Because of the symmetry in energy, the probability for 0 and 6ε are the same, 1ε and 5ε are the same ...

E = 0 1 state 0,0,0

E = 1 3 states (100),(010),(001)

E = 2 6 states (110),(101),(011),(200),(020),(002)

E = 3 7 states (111),(210),(120),(201),(102),(021),(012)

E	# states	Prob
0	1	1/27
1	3	3/27 = 1/9
2	6	6/27 = 2/9
3	7	7/27
4	6	2/9
5	3	1/9
6	1	1/27

Prob 4 At this temperature and pressure He behaves like an ideal gas. So we can use the Sackur-Tetrode equation, Eq (2.49), for the entropy. In addition ^3He has an extra 2^N states for every spatial configuration. From Eq. (2.45)

$$S_{^3He} = k \ln \Omega = S_{SacTet}(U,V,N) + k \ln 2^N = S_{SacTet}(U,V,N) + Nk\ln 2$$

$$S_{^3He} - S_{^4He} = Nk\ln 2 + S_{ST}(M_{^3He}) - S_{ST}(M_{^4He}) = Nk\ln 2 + \frac{3}{2}Nk\ln\left(\frac{M_{^3He}}{M_{^4He}}\right)$$

$$= Nk\ln 2 + \frac{3}{2}Nk\ln\left(\frac{3}{4}\right) = Nk\ln\left(2\left(\frac{3}{4}\right)^{3/2}\right) = 8.31\,\frac{J}{K}\,0.26 = 2.17\,\frac{J}{K}$$

Prob 5 This is like the many problems worked in class and homework. Get the multiplicity of each system. Get the constraint. Find the maximum multiplicity. This can also be done from entropy.

Eq. (3.28) $\quad S_a/k = N_a\ln N_a - N_2\ln N_2 - (N_a-N_2)\ln(N_a-N_2)$

Prob (3.25) $\quad S_b/k = q_b\ln(q_b+N_b) - q_b\ln q_b + N_b\ln(N_b+q_b) - N_b\ln N_b$

$$E = Qhf = q_b hf + N_2\phi hf \quad \Rightarrow \quad q_b = Q - \phi N_2 \qquad N_2 = \frac{Q-q_b}{\phi}$$

Find the max $\Rightarrow \frac{1}{k}\frac{\partial S_a}{\partial N_2} + \frac{1}{k}\frac{dq_b}{dN_2}\frac{\partial S_b}{\partial q_b} = 0$

$$0 = -\ln N_2 - \cancel{1} + \ln(N_a-N_2) + \cancel{1} + \frac{dq_b}{dN_2}\left(\ln(q_b+N_b) + \cancel{\frac{q_b}{q_b+N_b}} - \ln(q_b) - \cancel{1} + \cancel{\frac{N_b}{N_b+q_b}}\right)$$

$$\Rightarrow \ln\left(\frac{N_a-N_2}{N_2}\right) = \phi\ln\left(\frac{q_b+N_b}{q_b}\right) \quad \text{or} \quad \frac{\phi N_a - Q + q_b}{Q-q_b} = \left(\frac{q_b+N_b}{q_b}\right)^{\phi}$$

b) If $\phi = 1$ $\quad \frac{N_a - Q + q_b}{Q-q_b} = \frac{q_b+N_b}{q_b} \quad \Rightarrow \quad q_b(N_a-Q) + q_b^2 = -q_b^2 + (Q-N_b)q_b + QN_b$

$$2q_b^2 + (N_a+N_b-2Q)q_b - QN_b = 0 \qquad q_b^2 + \left(\frac{N}{2}-Q\right)q_b - \frac{QN_b}{2} = 0$$

$$q_b = \frac{Q-N/2}{2} \pm \frac{1}{2}\sqrt{(Q-N/2)^2 + 2QN_b}$$ The $-$ is not allowed because $q_b > 0$

Prob 6 a) Determine the entropy by finding the multiplicity. Not taking degeneracy into account, the number of ways for having 2 possible energies is

$$\Omega_{M_1=1, M_2=1} = \frac{N!}{N_1!\,N_2!}$$

Since there are M_1 possibilities when an atom has energy 0 and M_2 possibilities when an atom has energy ε, the actual multiplicity is

$$\Omega = M_1^{N_1} M_2^{N_2} \frac{N!}{N_1!\,N_2!} \overset{\text{Eq (2.14)}}{\cong} M_1^{N_1} M_2^{N_2} \frac{(N/e)^N}{(N_1/e)^{N_1}(N_2/e)^{N_2}} = N^N \left(\frac{M_1}{N-N_2}\right)^{N-N_2} \left(\frac{M_2}{N_2}\right)^{N_2}$$

$$S = k\ln\Omega = k\left[N\ln N + (N-N_2)\ln M_1 - (N-N_2)\ln(N-N_2) + N_2\ln(M_2) - N_2\ln(N_2)\right]$$

b) You can find T from $\frac{1}{T} \overset{\text{Eq 3.5}}{=} \left(\frac{\partial S}{\partial E}\right)_N \overset{E=N_2\varepsilon}{=} \frac{1}{\varepsilon}\left(\frac{\partial S}{\partial N_2}\right)_N$

$$\frac{1}{T} = \frac{k}{\varepsilon}\left[-\ln M_1 + \ln(N-N_2) + \frac{N-N_2}{N-N_2} + \ln(M_2) - \ln N_2 - \frac{N_2}{N_2}\right] = \frac{k}{\varepsilon}\ln\left[\frac{N-N_2}{N_2}\frac{M_2}{M_1}\right]$$

c) The probability for an atom to have energy ε is $\frac{N_2}{N}$

$$e^{\varepsilon/kT} = \left(\frac{N}{N_2} - 1\right)\frac{M_2}{M_1} \Rightarrow \frac{N}{N_2} = 1 + \frac{M_1}{M_2}e^{\varepsilon/kT} \Rightarrow \frac{N_2}{N} = \frac{1}{1+\frac{M_1}{M_2}e^{\varepsilon/kT}}$$

d) The change in entropy when adding one atom keeping E fixed is

$$\Delta S = \left(\frac{\partial S}{\partial N}\right)_{N_2}\Delta N^{\,\nearrow 1} = k\left[\ln N + \frac{N}{N} + \ln M_1 - \ln(N-N_2) - \frac{N-N_2}{N-N_2}\right] = k\ln\left(\frac{NM_1}{N-N_2}\right)$$