Chapter 7 Quantum Statistics

This chapter is mainly concerned with the case where the number of particles is comparable to or more than the number of states occupied. Situations where Fermions and Bosons behave differently.

In Chap 6 $Z_1 = \sum_s e^{-E(s)/kT}$ for 1 particle

If 2 particles in the same potential well $Z_{TOT} = \frac{1}{2} Z_1 Z_1$

This is qualitatively wrong when few states are occupied. For atomic gases this approx fails for $T \sim 1\,\mu K$. For electrons in a metal, fail at $T \sim 10^3 K$.

To see the problem look at 2 atoms in 3 states $Z_1 = 1 + e^{-\beta E_2} + e^{-\beta E_3}$

$$Z_{TOT} \stackrel{?}{=} \frac{1}{2} Z_1 Z_1 = \underbrace{\frac{1}{2}(1\cdot 1 + e^{-\beta E_2} e^{-\beta E_2} + e^{-\beta E_3} e^{-\beta E_3})}_{\text{wrong}} + 1 e^{-\beta E_2} + 1 e^{-\beta E_3} + e^{-\beta E_2} e^{-\beta E_3}$$

$$\text{Fermions } Z_{TOT} = 0 + 1 e^{-\beta E_2} + 1 e^{-\beta E_3} + e^{-\beta E_2} e^{-\beta E_3}$$

$$\text{Bosons } Z_{TOT} = (1\cdot 1 + e^{-\beta E_2} e^{-\beta E_2} + e^{-\beta E_3} e^{-\beta E_3}) + 1 e^{-\beta E_2} + 1 e^{-\beta E_3} + e^{-\beta E_2} e^{-\beta E_3}$$

Two Fermions can't be in the same state.
" Bosons can " " ".

There are two ways for dealing with this situation.

(1) Directly sum with correct boson/fermion factor for doubly occupied (theoretical machinery in QM raising lowering operators a, a^+ for bosons, b, b^+ for fermions)

(2) use a short cut based on the chemical potential

(1) and (2) give exactly the same answer. We'll do (2) by generalizing the formalism of Chap 6

Section 7.1 The Gibbs Factor

Treat quantum state S_a as the system. S_a is a single particle state (for example, the 18th state for a particle in a box). We will also let particles go into and out of state S_a

What is the probability the system has $N(S_a)$ particles in state S_a? Use thermo relations!

Like Chap 6 $\mathcal{P}(s_a) = \underset{\text{some const}}{C} \cdot \overset{\text{mult. of syst.}}{1} \cdot \underset{\text{mult. of reservoir}}{\Omega_{res}}(U - E(s_a), V, \underset{\text{number of objects in } s_a}{N - N(s_a)})$ $\quad U = E_{res} + E(s_a)$, $N = N_{res} + N(s_a)$

If the objects don't interact, $E(s_a) = N(s_a) \cdot \varepsilon_a$ $\quad \varepsilon_a$ = single particle energy in a

$$\frac{\mathcal{P}(s_a)}{\mathcal{P}(s_b)} = e^{[S_{res}(U-E(s_a), V, N-N(s_a)) - S_{res}(U-E(s_b), V, N-N(s_b))]/k}$$

Repeat idea of Chap 6 $S_{res}(U-E(s_a), V, N-N(s_a)) = S_{res}(U,V,N) - \frac{E(s_a)}{T} - \frac{-\mu}{T} N(s_a)$

$$\frac{\mathcal{P}(s_a)}{\mathcal{P}(s_b)} = \frac{e^{-\beta[E(s_a) - N(s_a)\mu]}}{e^{-\beta[E(s_b) - N(s_b)\mu]}}$$

Gibbs factor $= e^{-\beta[E(s) - \mu N(s)]}$

Grand Canonical Ensemble

In order for $\sum_s \mathcal{P}(s) = 1 \Rightarrow \mathcal{P}(s) = \frac{1}{\mathcal{Z}} e^{-\beta[E(s) - \mu N(s)]}$

$\mathcal{Z} = \sum_s e^{-\beta[E(s) - \mu N(s)]}$ is grand partition function

For more than 1 species, Gibbs factor $= e^{-\beta[E(s) - \mu_1 N_1(s) - \mu_2 N_2(s) \dots]}$

As noted above, often $E(s) \simeq \varepsilon(s) N(s)$ is a good approximation

Show $\bar{U} - \mu \bar{N} = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}$

Text book example: CO poisoning

A site on hemoglobin can bind O_2 or CO

only O_2 $\quad \mathcal{Z} = \overset{\text{no } O_2}{1} + e^{-\beta(\varepsilon - \mu)} \simeq 1 + 40$ $\quad \varepsilon \simeq -0.7\,eV$; $\mu = -kT \ln\left(\frac{V Z_{int}}{N v_Q}\right) \simeq -0.6\,eV$

$\mathcal{P}(\text{occupied } O_2) \simeq 40/41 \simeq 0.98$

$O_2 + \frac{1}{100}CO$ $\quad \mathcal{Z} = \overset{\text{no } O_2}{1} + \overset{O_2}{e^{-\beta(\varepsilon - \mu)}} + \overset{CO}{e^{-\beta(\varepsilon' - \mu')}}$ $\quad \varepsilon' \simeq -0.85\,eV$; $\mu' = \mu - kT \ln 100 \simeq -0.72\,eV$ (100 x less CO)

$\simeq 1 + 40 + 120$

$\mathcal{P}(\text{occupied } O_2) \simeq 40/161 \simeq 0.25$

As concentration of CO decreases μ' decreases which decreases $\mathcal{P}(CO)$

Prob 7.6 Show properties of $\mathcal{Z}$

Show $\bar{N} = kT \frac{\partial}{\partial \mu} \ln(\mathcal{Z}) = kT \frac{1}{\mathcal{Z}} \frac{\partial}{\partial \mu} \sum_s e^{-\beta[E(s)-\mu N(s)]} \overset{\checkmark}{=} \underbrace{kT\beta}_{=1} \sum_s \mathcal{P}(s) N(s)$

Show $\overline{N^2} = (kT)^2 \frac{1}{\mathcal{Z}} \frac{\partial^2}{\partial \mu^2} \mathcal{Z} = (kT)^2 \beta^2 \frac{1}{\mathcal{Z}} \sum_s N^2(s) e^{-\beta[E(s)-\mu N(s)]} \overset{\checkmark}{=} \overline{N^2}$

$\sigma_N^2 = \overline{N^2} - \bar{N}^2 = (kT)^2 \left[\frac{1}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial \mu^2} - \left(\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \mu} \right)^2 \right]$

But math gives

$\frac{\partial \bar{N}}{\partial \mu} = kT \frac{\partial}{\partial \mu} \left(\frac{1}{\mathcal{Z}} \frac{\partial \mathcal{Z}}{\partial \mu} \right) = kT \left(\frac{1}{\mathcal{Z}} \frac{\partial^2 \mathcal{Z}}{\partial \mu^2} - \frac{1}{\mathcal{Z}^2} \left(\frac{\partial \mathcal{Z}}{\partial \mu} \right)^2 \right) = \sigma_N^2 / kT \quad \Rightarrow \sigma_N = \sqrt{kT \frac{\partial \bar{N}}{\partial \mu}}$

Apply to ideal gas $\mu = -kT \left[\ln\left[\frac{V Z_{int}}{v_Q} \right] - \ln N \right] \Rightarrow \frac{\partial \mu}{\partial N} = \frac{kT}{N}$

$\Rightarrow \sigma_N = \sqrt{kT/(kT/N)} = \sqrt{N}$

Section 7.2 Bosons and Fermions

See book for 2 particles in 5 states

When the number of single particle states is much larger than the number of particles $Z_1 >> N$ no worries because the chance for two particles in the same state is small.

Example Ideal gas $V/N >> v_Q = \left(\frac{h^2}{2\pi m kT} \right)^{3/2}$ at room T, P $\Rightarrow$ no worries

Prob 7.10 5 particles in 1D harmonic oscillator (distinguishable, boson, fermion)

a) Ground state
- dist = 5, 0, 0, 0, 0, 0, ...
- boson = 5, 0, 0, 0, 0, 0, ...
- fermion = 1, 1, 1, 1, 1, 0, ...

b) hf above ground
- dist = 4, 1, 0, 0, 0, 0, ... 5 states
- boson = 4, 1, 0, 0, 0, 0, ... 1 state
- fermion = 1, 1, 1, 1, 0, 1, 1 state

					Total
c) $2hf$ above ground	dist = 4, 0, 1, 0, ... (5 states)	3, 2, 0, ... ($\frac{5 \cdot 4}{2} = 10$ states)			15 states
	boson = 4, 0, 1, 0, ... (1 state)	3, 2, 0, ... (1 state)			2 states
	fermion = 1, 1, 1, 1, 0, 0, 1, (1 state)	1, 1, 1, 0, 1, 1 (1 state)			2 states
$3hf$ above	dist 2, 3, 0.... (10 state)	3, 1, 1, (20 state)	4, 0, 0, 1 (5 state)		35 states
	bos " (1 state)	" (1 state)	" (1 state)		3 states
	fer 1, 1, 1, 1, 0, 0, 0, 1 (1 state)	1, 1, 1, 0, 1, 0, 1 (1 state)	1, 1, 0, 1, 1, 1 (1 state)		3 states

d) The occupied states are the same for dist. and bosons. The $e^{-\beta E}$ factors would also be the same. But the multiplicity for Boson excited states is much less than disting. $\Rightarrow$ Ground state is much more likely for Bosons than you might expect from the Boltzmann factors.

Distribution functions

Concentrate on one single particle state of a system (for example the $g=47$ state of harmonic oscillator). Follow the book notation $N \to n$. Focus on noninteracting $E = n\varepsilon$ $\quad$ ε = energy for 1 in state

$$\mathcal{P}(n) = \frac{1}{Z} e^{-n\beta(\varepsilon-\mu)}$$

Fermions Only $n=0$ or 1 allowed $\quad Z = 1 + e^{-\beta(\varepsilon-\mu)}$

The average number of Fermions in this state

$$\bar{n} = 0\,\mathcal{P}(0) + 1\,\mathcal{P}(1) = e^{-\beta(\varepsilon-\mu)}/(1+e^{-\beta(\varepsilon-\mu)}) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

$$\bar{n}_{FD}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}+1}$$

Bosons All n allowed $\quad n = 0, 1, 2, \ldots \quad Z = 1 + e^{-\beta(\varepsilon-\mu)} + e^{-2\beta(\varepsilon-\mu)} + \ldots$

$$Z = 1/[1-e^{-\beta(\varepsilon-\mu)}]$$

From Prob 7.6 $\quad \bar{n} = \frac{1}{\beta}\frac{1}{Z}\frac{\partial Z}{\partial \mu} = \frac{1}{\beta}[1-e^{-\beta(\varepsilon-\mu)}]\frac{\beta e^{-\beta(\varepsilon-\mu)}}{[1-e^{-\beta(\varepsilon-\mu)}]^2} = \frac{e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}}$

$$\bar{n}_{BE}(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)}-1}$$

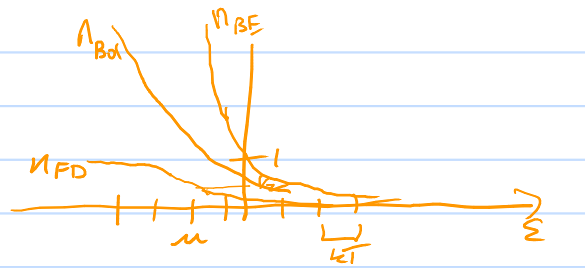
Distinguishable

$\mathcal{P} = \frac{1}{Z_1} e^{-\beta\varepsilon} \quad \Rightarrow \quad \bar{n}_{Bol} = \frac{N}{Z_1} e^{-\beta\varepsilon} \quad$ From Prob 6.44 $\quad \frac{N}{Z_1} = e^{\beta\mu}$

$$\bar{n}_{Bol} = \frac{1}{e^{\beta(\varepsilon-\mu)}+0}$$

Note $\quad \bar{n}_{FD}(\varepsilon) < \bar{n}_{Bol}(\varepsilon) < \bar{n}_{BE}(\varepsilon)$ for same μ

When $\beta(\varepsilon-\mu) >$ few $\quad \bar{n}_{FD} \approx \bar{n}_{Bol} \approx \bar{n}_{BE}$



Prob 7.14 $\frac{n_{BE}}{n_{FD}} = 1.01 = \frac{e^{\beta(\varepsilon-\mu)}+1}{e^{\beta(\varepsilon-\mu)}-1} = \frac{1+e^{-\beta(\varepsilon-\mu)}}{1-e^{-\beta(\varepsilon-\mu)}} \simeq 1+2e^{-\beta(\varepsilon-\mu)}$

$\beta(\varepsilon-\mu) = \ln 200 \simeq 5.3$ Since $\varepsilon > 0$ OK $-\beta\mu = \ln 200$

$\frac{V}{N}\frac{1}{v_Q} > 200 \Rightarrow \frac{kT}{P}\frac{1}{v_Q} > 200$ $T=300, P=1\,atm$ $\frac{kT}{P}\frac{1}{v_Q} \sim 10^8$

Prob 7.14 $N = \sum \bar{n}_{Bol} = \sum_s e^{-\beta E(s)} e^{+\beta\mu} = Z_1 e^{+\beta\mu} \Rightarrow \mu = -kT\ln\left(\frac{Z_1}{N}\right)$

Prob 7.12 Show prob. that $\varepsilon = \mu + x$ occupied equals prob $\varepsilon = \mu - x$ <u>not</u> occ.

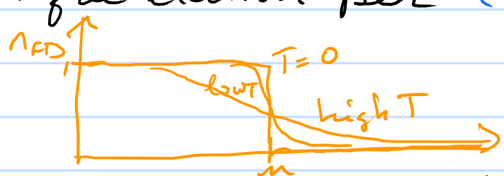
$\frac{1}{e^{\beta x}+1} \overset{?}{=} 1 - \frac{1}{e^{-\beta x}+1} = \frac{e^{-\beta x}}{e^{-\beta x}+1} \overset{\checkmark}{=} \frac{1}{1+e^{\beta x}}$

Section 7.3 Degenerate Fermi Gases

Electrons in a metal, protons/neutrons inside nucleus, electrons in white dwarf star, neutrons in neutron star

Example: Ideal gas when $\frac{V}{N} \ll v_Q$ (Electrons at 300 K have $v_Q \simeq (4\,nm)^3$ but in a metal there's 1 free electron per $\sim(0.2\,nm)^3$)

Examine n_{FD} vs ε for diff T



Notice at $T=0$, n_{FD} is a step function at $\varepsilon = \mu$. Why? At $T=0$ the system is in its ground state. Order the states by energy. There are N particles $\Rightarrow$ the first N states have occupancy 1 and all others are unoccupied.

The energy $\varepsilon_N = \mu(T=0) = \varepsilon_F$ = the Fermi energy

When a gas of Fermions mostly occupy states with $\varepsilon < \varepsilon_F$, then called Degenerate Fermi Gas

Notice $\left(\frac{\partial U}{\partial N}\right)_{S,V} = \mu$ at $T=0$ $U(N+1) = U(N) + \varepsilon_F \overset{\checkmark}{=} U(N) + \mu$

Non interacting spin 1/2

Fermions in a box This is a simple example, but it contains the trends of real cases.

$E = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2) \Rightarrow \varepsilon_F = \frac{h^2 n_{max}^2}{8mL^2}$ $\quad (n_x = 1, 2, 3\ldots,\ n_y = 1, 2, 3, \ldots \ldots)$

But this doesn't help unless you know how n_{max} depends on N

Think of $n_x^2 + n_y^2 + n_z^2$ as spherical radius.

$N = \#$ of states with $n_x^2 + n_y^2 + n_z^2 < n_{max}^2$

This is like finding volume of 1/8 of sphere

For spin 1/2, every n_x, n_y, n_z is 2 states (↑ and ↓)

$N = 2 \times \frac{1}{8}\frac{4\pi}{3} n_{max}^3 \Rightarrow n_{max} = (3N/\pi)^{1/3}$

Compute Fermi energy $\varepsilon_F = \frac{h^2}{8mL^2}\left(\frac{3N}{\pi}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{L^3}\right)^{2/3} = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3}$

generalize to non cube box

Compute U $\quad U = 2\sum_{n_x n_y n_z}\frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2) = 2\frac{h^2}{8mL^2}\frac{1}{8}\int_0^{n_{max}} n^2\, 4\pi n^2\, dn$

$= \frac{h^2}{8mL^2}\pi\frac{1}{5}n_{max}^5 = \frac{h^2 n_{max}^2}{8mL^2}\frac{1}{5}\pi n_{max}^3 = \varepsilon_F \frac{3}{5}N$

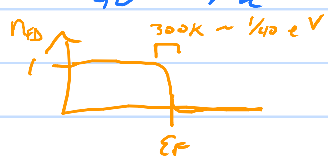
The average energy per particle is $U/N = \frac{3}{5}\varepsilon_F$

Example Estimate for metal $\frac{N}{V} \sim \frac{1}{(0.2\times 10^{-9}\text{m})^3} = 1.25\times 10^{29}\frac{1}{\text{m}^3}$

$\varepsilon_F = \frac{(6.626\times 10^{-34}\text{Js})^2}{8\ \ 9.11\times 10^{-31}\text{kg}}\left(\frac{3}{\pi}1.25\times 10^{29}\frac{1}{\text{m}^3}\right)^{2/3} = 1.46\times 10^{-18}\text{J} \sim 9\text{eV} \sim 10^5\text{K}\ k_B$

Room temperature $\quad 300\text{K} \sim \frac{1}{40}\text{eV}/k \Rightarrow \varepsilon_F \gg kT$

$\Rightarrow$ For electrons in metal



The degeneracy pressure $\quad P = -\left(\frac{\partial U}{\partial V}\right)_{S,N} = -\frac{3}{5}N\frac{\partial \varepsilon_F}{\partial V} = -\frac{3}{5}N\left(-\frac{2}{3}\right)\frac{\varepsilon_F}{V} = \frac{2}{3}\frac{U}{V}$

This is mostly canceled by the attractive electrostatic forces.

The Bulk modulus $\quad B = -V\left(\frac{\partial P}{\partial V}\right)_T = -V\frac{\partial}{\partial V}\left(\frac{C}{V^{5/3}}\right) = \frac{5}{3}\frac{C}{V^{5/3}} = \frac{5}{3}\frac{2}{3}\frac{U}{V} = \frac{10}{9}\frac{U}{V}$

Prob 7.20 $\quad \frac{N}{V} = 10^{32}\frac{1}{\text{m}^3} \qquad T = 10^7\text{K}$

$\varepsilon_F = \frac{(6.626\times 10^{-34}\text{Js})^2}{8\ \ 9.11\times 10^{-31}}\left(\frac{3}{\pi}10^{32}\frac{1}{\text{m}^3}\right)^{2/3} = 1.26\times 10^{-16}\text{J} \sim 800\text{eV} \sim 9\times 10^6\text{K}\cdot k_B$

Since $\varepsilon_F \sim kT$, the electron gas is neither degenerate nor nearly classical

For protons $\quad m_p \sim 1840\, m_e \Rightarrow \varepsilon_{F,prot} \sim \varepsilon_{F,elec}/1840 \Rightarrow \varepsilon_{F,prot} \ll kT$ classical

Prob 7.23 White dwarf (show that electron degeneracy pressure can prevent star collapse

a) Compute gravitational PE. Imagine assembling from shell

For analytic, approximate as const density

$\Delta U = -\frac{Gm}{r}\,dm = -\frac{G}{r}\left(\rho\frac{4}{3}\pi r^3\right)\left(\rho 4\pi r^2 dr\right)$

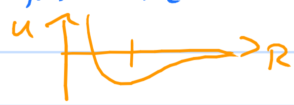
$U_g = \int_0^R -G\rho^2\frac{(4\pi)^2}{3} r^4\,dr = -G\rho^2\frac{(4\pi)^2}{3}\frac{1}{5}R^5 = -\frac{3}{5}\frac{GM^2}{R}$ scaling right, 3/5 probably off

b) Guess that most of the KE is from nonrelativistic electrons

$U_{KE} = \frac{3}{5}N\varepsilon_F = \frac{3}{5}\frac{M}{(2m_p)}\frac{h^2}{8m_e}\left(\frac{3}{\pi}\frac{M}{2m_p}\frac{1}{\frac{4\pi}{3}R^3}\right)^{2/3} = \left(\frac{3}{80}\left(\frac{9}{8\pi^2}\right)^{2/3}\right)h^2\left(\frac{M}{m_p}\right)^{5/3}\frac{1}{m_e}\frac{1}{R^2}$

$= 0.0088\left(\frac{M}{m_p}\right)^{5/3}\frac{h^2}{m_e R^2}$

c) $U = U_g + U_{KE}$ has minimum



$\frac{dU}{dR} = 0 = \frac{3}{5}\frac{GM^2}{R^2} - 0.0088\left(\frac{M}{m_p}\right)^{5/3}2\frac{h^2}{m_e R^3}$

$\Rightarrow R_{min} = \frac{10}{3}\,0.0088\,\frac{1}{G}\frac{h^2}{m_e m_p^{5/3}}\frac{1}{M^{1/3}} = 0.029\frac{h^2}{G m_e m_p^{5/3}}\frac{1}{M^{1/3}}$

Notice R_{min} decreases with $M^{1/3}$

d) Evaluate R_{min} for mass of sun $R_{min} = 0.029\frac{(6.626\times10^{-34}Js)^2}{6.673\times10^{-11}m^3kg^{-1}s^{-2}(2\times10^{30}kg)^{1/3}\,1.67\times10^{-27}kg \times 9.11\times10^{-31}}$

$R_{min} = 7.2\times10^6 m = 7200$ km

$\rho = 2\times10^{30}kg \,/ \left(\frac{4\pi}{3}(7.2\times10^6 m)^3\right) = 1.3\times10^9\frac{kg}{m^3} \sim 1.3\times10^6 \times$ denser than water

e) $\varepsilon_F = \frac{h^2}{8m_e}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3} = \frac{h^2}{8m_e}\left(\frac{3}{\pi}\frac{\rho}{2m_p}\right)^{2/3} = \frac{(6.6\times10^{-34}Js)^2}{8\cdot 9.11\times10^{-31}kg}\left(\frac{3}{\pi}\frac{1.3\times10^9 kg/m^3}{2\cdot1.67\times10^{-27}kg}\right)^{2/3} = 3.1\times10^{-14}J$

$\Rightarrow T_F = \varepsilon_F/k = 2.3\times10^9 K \gg$ temperature

f) From prob 7.22 $U_{KE} = \frac{3}{4}N\varepsilon_F = \frac{3}{4}N\frac{hc}{2}\left(\frac{3}{\pi}\frac{N}{V}\right)^{1/3} = \frac{3}{8}hc\left(\frac{M}{2m_p}\right)^{4/3}\left(\frac{9}{4\pi^2R^3}\right)^{1/3}$

$= 0.091\,hc\left(\frac{M}{m_p}\right)^{4/3}1/R$

$\Rightarrow U = U_g + U_{KE} = \left[0.091\,hc\left(\frac{M}{m_p}\right)^{4/3} - \frac{3}{5}GM^2\right]/R$

If [ ] < 0 then the minimum is $R = 0$

If [ ] > 0 then R increases until substantially nonrelativistic

g) The break even [ ] = 0 $\quad 0.091\,hc\left(\frac{M}{m_p}\right)^{4/3} = \frac{3}{5}GM^2$ implies

collapse for $M > \left[0.091\,hc\,\frac{5}{3G}\right]^{3/2}\frac{1}{m_p^2} = 3.4\times10^{30}$ kg

Some considerations relativistic $\varepsilon_F \sim m_e c^2 \sim 8.2\times10^{-14}$ J

The Fermi energy for sun is less than mc^2

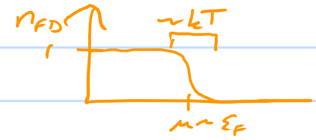
Best modern calculation ~ stability mass ~ 1.4 solar masses ~ 3×10^{30} kg

What happens? Star collapses until lower energy $e + p \to n + \nu_e$

Nova → neutron star (neutron is spin ½ Fermion)

In neutron star, the neutron degeneracy pressure prevents collapse.

Unless... M is so big neutrons are relativistic which can't prevent further collapse → black hole



Now investigate small T, not $T=0$

How does μ change with T?

Easier to determine how U increases with T

Number of affected particles $\sim N\frac{kT}{\varepsilon_F}$; Energy acquired $\sim kT$

$$\Rightarrow U = \frac{3}{5}N\varepsilon_F + \text{const } N\frac{k^2T^2}{\varepsilon_F} = \frac{3}{5}N\varepsilon_F + \frac{\pi^2}{4}N\frac{k^2T^2}{\varepsilon_F}$$

$$C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{\pi^2}{2}Nk^2\frac{T}{\varepsilon_F}$$

Prob 7.25 $\frac{C_V}{Nk} \simeq \frac{\pi^2}{2}\frac{kT}{\varepsilon_F} \simeq \frac{\pi^2}{2}\frac{\frac{1}{40}\,eV}{7\,eV} \sim 0.017$

Compare to vibrations $C_V/Nk \sim 3$

$\Rightarrow$ Electrons contribute $\sim \frac{1}{2}\%$ to heat capacity at room temp.

We'll find that the proportions swap at low T

Density of states $g(\varepsilon)\Delta\varepsilon$ = number of states between ε and $\varepsilon + \Delta\varepsilon$

Example: particle in box $U = 2\,\frac{1}{8}\,4\pi\int_0^\infty n_{FD}(\varepsilon(n))\,\varepsilon(n)\,n^2\,dn$ $\quad \varepsilon(n) = \frac{h^2n^2}{8mL^2}$

Want to turn this into an integral over energy

$n = \sqrt{\frac{8mL^2}{h^2}}\sqrt{\varepsilon} \qquad dn = \frac{1}{2}\sqrt{\frac{8mL^2}{h^2}}\frac{1}{\sqrt{\varepsilon}}\,d\varepsilon$

$$\Rightarrow U = \int_0^\infty n_{FD}(\varepsilon)\,\varepsilon\,\pi\,\frac{8mL^2}{h^2}\varepsilon\,\frac{1}{2}\sqrt{\frac{8mL^2}{h^2}}\frac{1}{\sqrt{\varepsilon}}\,d\varepsilon \equiv \int_0^\infty n_{FD}(\varepsilon)\,\varepsilon\,g(\varepsilon)\,d\varepsilon$$

$g(\varepsilon) = \frac{\pi}{2}\left(\frac{8mL^2}{h^2}\right)^{3/2}\sqrt{\varepsilon} = \frac{\pi}{2}(8m)^{3/2}\frac{V}{h^3}\sqrt{\varepsilon}$ $\quad$ units 1/energy

For $T=0$ $\quad N = \int_0^{\varepsilon_F} g(\varepsilon)\,d\varepsilon = \frac{\pi}{2}(8m)^{3/2}\frac{V}{h^3}\frac{2}{3}\varepsilon_F^{3/2}$ $\quad$ Eq 7.39 $\quad \varepsilon_F = \frac{h^2}{8m}\left(\frac{3}{\pi}\frac{N}{V}\right)^{2/3}$

For $T=0$ $\quad U = \int_0^{\varepsilon_F}\varepsilon\,g(\varepsilon)\,d\varepsilon = \frac{\pi}{2}(8m)^{3/2}\frac{V}{h^3}\frac{2}{3}\varepsilon_F^{3/2}\frac{3}{5}\varepsilon_F = \frac{3}{5}N\varepsilon_F$ $\quad$ Eq 7.42

How to deal with small T (not $T=0$) due to Sommerfeld

$$N = \int_0^\infty g(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon \quad \text{and} \quad U = \int_0^\infty \varepsilon\,g(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon$$

Use the equation for N to determine μ. Use that μ to determine U. μ decreases with T because $g(\varepsilon)$ is an increasing function of ε (n_{FD} is smaller for $\varepsilon<\mu$ and larger for $\varepsilon>\mu$).

U increases with T because particles reach larger ε

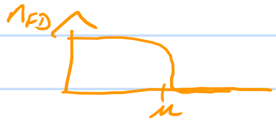
The following is a math tangent for a general treatment of N and U integrals.

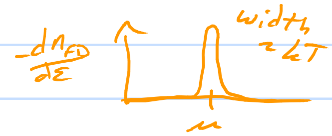
Do the integral $A \equiv \int_0^\infty \frac{dF(\varepsilon)}{d\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\,d\varepsilon \cong F(\mu) + \frac{\pi^2}{6}(kT)^2 F''(\mu)$

Integrate by parts

$$A = \underbrace{\cancel{F(\varepsilon)\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\Big|_0^\infty}}_{\to 0} + \int_0^\infty F(\varepsilon)\left[-\frac{d}{d\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}+1}\right]d\varepsilon \qquad F(0)=0,\ n_{FD}(\infty)=0$$

$-\frac{d}{d\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}+1}$ is strongly peaked at $\varepsilon=\mu$





$$-\frac{d}{d\varepsilon}\frac{1}{e^{\beta(\varepsilon-\mu)}+1} = \beta\frac{e^{\beta(\varepsilon-\mu)}}{(e^{\beta(\varepsilon-\mu)}+1)^2}$$

Change variables $x = \beta(\varepsilon-\mu) \Rightarrow \varepsilon = \mu + kTx$

$$\Rightarrow A = \int_{-\mu/kT}^{\infty} F(\mu+kTx)\,\cancel{\beta}\frac{e^x}{(e^x+1)^2}\cancel{kT}\,dx$$

Now make approximations

Small $T \Rightarrow \mu/kT \gg 1 \Rightarrow e^{-\mu/kT} \simeq 0 \Rightarrow$ Ok for lower limit $\to -\infty$

Also since $kT \ll \mu$, OK to do Taylor series for F

$$F(\mu+kTx) = F(\mu) + kT\,F'(\mu)\,x + \frac{(kT)^2}{2}F''(\mu)\,x^2 + \dots$$

One last thing to note $\frac{e^x}{(e^x+1)^2} = \frac{1}{(e^{x/2}+e^{-x/2})^2}$ is symmetric in x

Prob 7.30 Show A only has even powers of T

$$A = \int_{-\infty}^{\infty}\left[F(\mu) + kT\,F'(\mu)\,x + \frac{(kT)^2}{2}F''(\mu)\,x^2 + \dots\right]\frac{1}{(e^{x/2}+e^{-x/2})^2}\,dx$$

All odd powers of x integrate to 0

$$\Rightarrow A = F(\mu)\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}dx + \frac{(kT)^2}{2}F''(\mu)\int_{-\infty}^{\infty}x^2\frac{e^x}{(e^x+1)^2}dx$$

The two integrals are numbers (don't depend on F, T, ...)

$$\int_{-\infty}^{\infty}\frac{e^x}{(e^x+1)^2}dx = -\frac{1}{e^x+1}\Big|_{-\infty}^{\infty} = -\frac{1}{e^\infty+1} - -\frac{1}{e^{-\infty}+1} = 1$$

$$\int_{-\infty}^{\infty}x^2\frac{e^x}{(e^x+1)^2}dx = 2\int_0^\infty x^2\frac{e^x}{(e^x+1)^2}dx = 2\int_0^\infty -\frac{d}{dx}\left(\frac{x^2}{e^x+1}\right) + \frac{2x}{e^x+1}\,dx = 4\int_0^\infty \frac{x}{e^x+1}dx$$

$$= 2\underbrace{\zeta(2)}_{\text{Rieman zeta fct}} = \pi^2/3$$ Eqs B.35 and B.45

$$A \cong F(\mu) + \frac{\pi^2}{6}(kT)^2F''(\mu) + (\ )(kT)^4$$

Use this to evaluate for $N = \int g(\varepsilon)\, n_{FD}(\varepsilon)\, d\varepsilon \Rightarrow \frac{dF}{d\varepsilon} = g(\varepsilon) = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \sqrt{\varepsilon}$

$F(\varepsilon) = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{2}{3}\varepsilon^{3/2}$

$N = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{2}{3}\mu^{3/2} + \frac{\pi^2}{6}(kT)^2 \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{1}{2\mu^{1/2}}$ a horrible equation

Want to use this to determine $\mu(T)$. Note $\frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{2}{3} = N/\varepsilon_F^{3/2}$

$\Rightarrow N = N\left(\mu/\varepsilon_F\right)^{3/2} + \frac{\pi^2}{6}(kT)^2 \frac{3}{4}\frac{N}{\mu^{1/2}\varepsilon_F^{3/2}}$

$1 = \left(\mu/\varepsilon_F\right)^{3/2} + \frac{\pi^2}{8}(kT)^2\frac{1}{\mu^{1/2}\varepsilon_F^{3/2}} \Rightarrow \left(\frac{\mu}{\varepsilon_F}\right) = \left[1 - \frac{\pi^2}{8}(kT)^2\frac{1}{\mu^{1/2}\varepsilon_F^{3/2}}\right]^{2/3}$

Since $\mu \simeq \varepsilon_F$ and $(kT/\varepsilon_F)^2 \ll 1$

$\mu/\varepsilon_F \cong 1 - \frac{\pi^2}{8}\left(\frac{kT}{\varepsilon_F}\right)^2\frac{2}{3} = 1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2$ decreases with T

Now evaluate $U = \int \varepsilon\, g(\varepsilon)\, n_{FD}(\varepsilon)\, d\varepsilon$ $\quad \frac{dF}{d\varepsilon} = \varepsilon g(\varepsilon) = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \varepsilon^{3/2}$

$F = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{2}{5}\varepsilon^{5/2}$

$U = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V \frac{2}{5}\mu^{5/2} + \frac{\pi^2}{6}(kT)^2\frac{(8m)^{3/2}}{h^3} V \frac{3}{2}\mu^{1/2}$

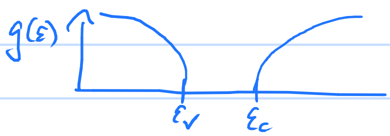
$= \frac{3}{5} N \frac{\mu^{5/2}}{\varepsilon^{3/2}} + \frac{\pi^2}{6}(kT)^2 \frac{9}{4} N \frac{\mu^{1/2}}{\varepsilon^{3/2}}$ Plug in $\mu = \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right]$

$= \frac{3}{5} N \varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right]^{5/2} + \frac{3\pi^2}{8}\left(\frac{kT}{\varepsilon_F}\right)^2 N\varepsilon_F\left[1 - \frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right]^{1/2}$

$= \frac{3}{5} N\varepsilon_F + \frac{3}{5}N\varepsilon_F\left(-\frac{5}{2}\frac{\pi^2}{12}\left(\frac{kT}{\varepsilon_F}\right)^2\right) + \frac{3\pi^2}{8}\left(\frac{kT}{\varepsilon_F}\right)^2 N\varepsilon_F = \frac{3}{5}N\varepsilon_F + \frac{\pi^2}{4}N\varepsilon_F\left(\frac{kT}{\varepsilon_F}\right)^2$

This matches the handwaving argument from the start of the section!

Prob 7.33



For $\varepsilon < \varepsilon_v$ $\quad g(\varepsilon) = g_0(\varepsilon_v - \varepsilon)^{1/2}$
$\varepsilon > \varepsilon_c$ $\quad g(\varepsilon) = g_0(\varepsilon - \varepsilon_c)^{1/2}$ $\quad g_0 = \frac{\pi}{2}\frac{(8m)^{3/2}}{h^3} V$

a) $N = \int g(\varepsilon)\, n_{FD}(\varepsilon)\, d\varepsilon$

Remember n_{FD} is symmetrical in the sense the decrease at an $\varepsilon < \mu$ is exactly the increase at the corresponding $\varepsilon > \mu$

Since $g(\varepsilon)$ is symmetric about $\varepsilon = \frac{\varepsilon_v + \varepsilon_c}{2} \Rightarrow \mu = \frac{\varepsilon_v + \varepsilon_c}{2}$

b) The number in the conduction band is

$N_c = \int_{\varepsilon_c}^{\infty} g(\varepsilon)\, n_{FD}(\varepsilon)\, d\varepsilon = g_0 \int_{\varepsilon_c}^{\infty}\sqrt{\varepsilon - \varepsilon_c}\, n_{FD}(\varepsilon)\, d\varepsilon = g_0\int_0^{\infty}\sqrt{\varepsilon}\, n_{FD}(\varepsilon + \varepsilon_c)\, d\varepsilon$

The integral can't be done analytically but approximations work well when $\varepsilon_c - \varepsilon_v \gg kT$

$n_{FD}(\varepsilon + \varepsilon_c) = \frac{1}{e^{\beta(\varepsilon_c - \mu)}e^{\beta\varepsilon} + 1} \simeq e^{-\beta(\varepsilon_c - \mu)}e^{-\beta\varepsilon} \Rightarrow N_c \cong g_0 e^{-\beta(\varepsilon_c - \mu)}\int_0^{\infty}\sqrt{\varepsilon}\, e^{-\beta\varepsilon}\, d\varepsilon$

Change variables $\varepsilon = x^2$

$N_c = g_0 e^{-\beta(\varepsilon_c - \varepsilon_v)/2}\int_0^{\infty} x\, e^{-\beta x^2}\, 2x\, dx = g_0 e^{-\beta(\varepsilon_c - \varepsilon_v)/2}\frac{1}{2}\sqrt{\frac{\pi}{\beta^3}} = \frac{\pi}{2}\frac{(8mkT)^{3/2}}{h^3}\frac{\sqrt{\pi}}{2} V e^{-\frac{(\varepsilon_c - \varepsilon_v)}{2kT}}$

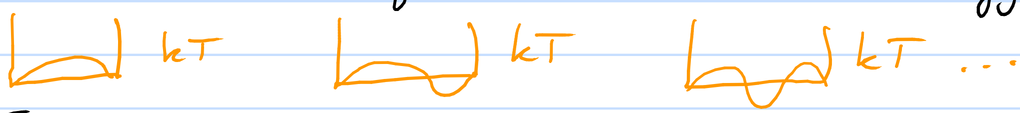
$= 2\frac{V}{v_Q} e^{-(\varepsilon_c - \varepsilon_v)/2kT}$ $\quad v_Q = \left(\frac{h}{\sqrt{2\pi mkT}}\right)^3$ Eq 7.18

Our approx works when $e^{-(\varepsilon_c - \varepsilon_v)/2kT}$ is small. For Si $\varepsilon_c - \varepsilon_v = 1.11\ eV$

$kT = \frac{1}{40}\ eV \Rightarrow e^{-22.2} = 2.3\times 10^{-10}$. Note how changes with T

Section 7.4 Blackbody Radiation

This treats the thermodynamics of the electromagnetic field.
First QM effects discovered using this!!!
Classical E+M plus thermo $\rightarrow$ ultraviolet catastrophe. Every classical harmonic oscillator mode has $\frac{1}{2}kT + \frac{1}{2}kT$ energy.
There are an ∞ of modes $\Rightarrow$ ∞ E+M energy in a cavity.



From QM, if $hf \ll kT$, then should get classical result. If $hf > kT$, results wildly different.

Classically, each mode behaves like a harmonic oscillator.
QM $\rightarrow$ energies are quantized $E_n = 0\,hf,\ 1\,hf,\ 2\,hf \ldots$

Can find $\bar{E}$ in a mode using $\bar{E} = -\frac{1}{Z}\frac{\partial Z}{\partial \beta}$ $\quad Z = 1 + e^{-\beta hf} + e^{-2\beta hf} + \ldots$

$Z = 1/(1-e^{-\beta hf}) \Rightarrow \bar{E} = -(1-e^{-\beta hf})^{-}hf\,e^{-\beta hf}/(1-e^{-\beta hf})^2$

$\bar{E} = hf/(e^{\beta hf}-1)$

If $\quad hf \ll kT \quad \bar{E} = hf/(1+\beta hf - 1) = kT$ ✓

$\quad\quad hf \gg kT \quad \bar{E} = hf\,e^{-\beta hf}$

Note $\bar{E} = hf\,\bar{n}_{pl} \Rightarrow \bar{n}_{pl} = \frac{1}{e^{\beta hf}-1} = \frac{1}{e^{\beta\varepsilon}-1}$

Interpretation is $E_n = 0\,hf,\ 1\,hf,\ 2\,hf, \ldots n\,hf \ldots$ where n is the number of photons, $\Rightarrow$ the photons are bosons with chemical potential $\mu = 0$ Why?

Now we need to figure out how to count the modes
Suppose you have a cube, the modes can be indexed by n_x, n_y, n_z
$\sin(\frac{\pi n_x}{L}x)\sin(\frac{\pi}{L}n_y y)\sin(\frac{\pi}{L}n_z z) \quad n_x = 1, 2, \ldots; n_y = 1, 2, \ldots \ldots$
Remember from E+M $k = \sqrt{k_x^2+k_y^2+k_z^2} \quad k_x = \frac{\pi n_x}{L}, k_y = \ldots$
and $\omega = kc \Rightarrow f = kc/2\pi$

From QM $\varepsilon_{n_x n_y n_z} = hf = \frac{hc}{2L}\sqrt{n_x^2+n_y^2+n_z^2} = \frac{hc}{2L}n \quad n \equiv \sqrt{n_x^2+n_y^2+n_z^2}$
Just like the ideal gas we can get U by converting sums to integrals

$$U = \overset{\text{polarization}}{2}\sum_{n_x n_y n_z}\frac{hc}{2L}\sqrt{n_x^2+n_y^2+n_z^2}\;\bar{n}_{pl}\left(\frac{hc}{2L}\sqrt{n_x^2+n_y^2+n_z^2}\right) = \frac{hc}{L}\frac{1}{8}4\pi\int_0^\infty n\,\frac{1}{e^{\frac{\beta hc}{2L}n}-1}\,n^2\,dn$$

Convert the n integral to ε integral $\quad n = \frac{2L}{hc}\varepsilon$

$$U = \frac{hc}{2L}\pi\left(\frac{2L}{hc}\right)^4 \int_0^\infty \frac{\varepsilon^3}{e^{\beta\varepsilon}-1}\,d\varepsilon \implies \frac{U}{V} = \frac{8\pi}{(hc)^3}\int_0^\infty \frac{\varepsilon^3}{e^{\beta\varepsilon}-1}\,d\varepsilon$$

← no longer depend on shape

$\implies \frac{8\pi}{(hc)^3}\frac{\varepsilon^3}{e^{\beta\varepsilon}-1}\Delta\varepsilon$ is the energy density of photons between ε and $\varepsilon+\Delta\varepsilon$

Change variables $\quad \varepsilon = kTx$

$$\frac{U}{V} = \frac{8\pi}{(hc)^3}(kT)^4\int_0^\infty \frac{x^3}{e^x-1}\,dx \implies \frac{U}{V} \propto T^4 \quad (\text{not } \infty)$$

Appendix A $\int_0^\infty \frac{x^3}{e^x-1}\,dx = \frac{\pi^4}{15} \implies \frac{U}{V} = \frac{8\pi^5}{15(hc)^3}(kT)^4$

Prob 7.37 max of $\frac{x^3}{e^x-1}$

$$\frac{d}{dx}\left(\frac{x^3}{e^x-1}\right) = \frac{3x^2}{e^x-1} - \frac{x^3e^x}{(e^x-1)^2} = 0 \implies 3(e^x-1) = xe^x \implies e^{-x} = 1-\frac{x}{3}$$

$e^{-2.82} = 0.0596 \quad 1-\frac{2.82}{3} = 0.0600$

Prob 7.38 How does $U(\varepsilon)$ change with T?

$U(\varepsilon) = \frac{8\pi}{(hc)^3}(kT)^3\frac{(\varepsilon/kT)^3}{e^{\varepsilon/kT}-1}$ $\quad$ height $\propto T^3$ $\quad$ position of max $\varepsilon \propto T$

Example What's the total energy in cavity from photons compared to 1 mole monatomic gas? $V = 1\,m^3$ $\quad T = 300\,K$

$U = \frac{3}{2}NkT = \frac{3}{2}RT = 3.7\,kJ$

$U = 1\,m^3\,\frac{8\pi^5}{15(6.6\times10^{-34}\,3\times10^8)^3}(1.38\times10^{-23}\,300)^4 = 6.2\times10^{-6}\,J$ $\quad$ much less!

What is heat capacity? $\quad C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{32\pi^5}{15}\left(\frac{k_BT}{hc}\right)^3 V k_B$

What is entropy? $\quad S = \int_0^T \frac{C_V(T)}{T}dT = \frac{32}{45}\pi^5\left(\frac{k_BT}{hc}\right)^3 V k_B = \frac{4}{3}\frac{U}{T}$

Evaluate $1\,m^3$, $300\,K$ $\quad S = \frac{4}{3}\frac{6.2\times10^{-6}\,J}{300\,K} = 2.7\times10^{-8}\,J/K$ $\quad$ very small

The microwave background of the universe looks like blackbody radiation at 2.735 K! Look at fit Fig 7.20
Differences from fit are now used to understand early universe cosmology

Prob 7.45 What pressure is exerted on walls from photons?

$P = -\left(\frac{\partial U}{\partial V}\right)_{S,N}$ Problem $S(T) = \frac{32\pi^5}{45} V k_B \left(\frac{k_B T}{hc}\right)^3$ => as you change V, you must also change T to keep $S = \text{const}$ (Don't need to worry about N because $\cancel{\mu\, dN} \to 0$)

Use compact notation $S \equiv a \cdot V \cdot T^3$ $\quad U \equiv b \cdot V \cdot T^4$

$\Rightarrow T = \left(S/aV\right)^{1/3} \quad U = b\cdot\left(S/a\right)^{4/3} V^{-1/3}$

$$P = -\frac{\partial U}{\partial V} = \frac{1}{3} b \left(\frac{S}{a}\right)^{4/3} V^{-4/3} = \frac{1}{3}\frac{U}{V} = \frac{8\pi^5}{45}\left(\frac{kT}{hc}\right)^3 kT$$

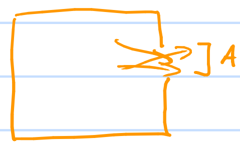
At 1500 K $P = \frac{8\pi^5}{45}\left(\frac{1.38\times10^{-23}\; 1500}{6.6\times10^{-34}\; 3\times10^{8}}\right)^3 1.38\times10^{-23}\, 1500\ \text{Pa} = 0.0013\ \text{Pa}$

Factor of $\sim10^8$ smaller than 1 atm. At 300 K, $P \to \frac{1}{5^4}$ smaller

At the center of the sun, $T = 1.5\times10^7$ K $\Rightarrow P = 0.0013\ \text{Pa}\cdot\left(10^4\right)^4 = 1.3\times10^{13}\ \text{Pa}$

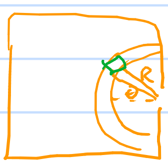
From electrons and protons $P = 2\cdot\frac{1.5\times10^5\ \text{kg/m}^3}{1.67\times10^{-27}\ \text{kg}}\, 1.38\times10^{-23}\,\frac{\text{J}}{\text{K}}\, 1.5\times10^7\ \text{K} = 3.7\times10^{16}\ \text{Pa}$

Photons contribute ~ 1/1000 of pressure



An important idea is the power escaping through a hole of area A. ($\sqrt{A}$ needs to be substantially larger than average wavelength.) Why?

Before doing any math, we know $\frac{\text{Power}}{\text{area}} \propto \frac{U}{V} c = \text{const}\ \frac{(kT)^4}{h^3 c^2}$



The energy in the little volume

$$E = \frac{U}{V} R^2 \sin\theta\, d\theta\, d\phi\, dR = \frac{U}{V} R^2 \sin\theta\, d\theta\, d\phi\, c\, \delta t$$

The probability that the energy in that volume will go directly through the hole is $A\cos\theta\,/(4\pi R^2)$

$$\Rightarrow \text{Power}/A = \frac{U}{V} c\, \cancel{R^2} \sin\theta\, d\theta\, d\phi \cos\theta\,/4\pi \cancel{R^2} = \frac{Uc}{V4\pi}\cos\theta \sin\theta\, d\theta\, d\phi$$

Integrate over θ, ϕ $\quad \frac{\text{Power}}{A} = \frac{Uc}{V4\pi}\int_0^{\pi}\left[\int_{-\pi/2}^{\pi/2} \cos\theta \sin\theta\, d\theta\right] d\phi = \frac{Uc}{V4\pi}\,\pi\, \frac{1}{2}\sin^2\theta\Big|_{-\pi/2}^{\pi/2}$

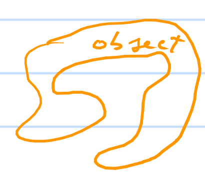
$$= \frac{1}{4} c\, U/V$$

$$= \frac{2\pi^5}{15}\frac{(kT)^4}{h^3c^2} = \sigma T^4$$

σ = Stefan-Boltzmann const $= 5.67\times10^{-8}\ \frac{\text{W}}{\text{m}^2\text{K}^4}$

Radiation from object $\quad \text{Power} = \sigma \underbrace{e}_{\text{emissivity}} \underbrace{A}_{\text{surface area of object}} T^4$

Not for



$0 < e < 1$ $\quad e$ = fraction of photons absorbed/emitted

$e = 1$ black body

$e = 0$ perfectly reflecting (doesn't emit!)

The solar radiation at earth orbit is $1370\ \mathrm{W/m^2}$. What is T_{sun}?

$$P_{sun} = P_{earth} \frac{4\pi R_{s-e}^2}{4\pi R_s^2} = 1370 \frac{W}{m^2}\left(\frac{1.5\times10^{11} m}{7.0\times10^{8} m}\right)^2 = 6.3\times10^7 \frac{W}{m^2}$$

$$\Rightarrow T = \left(6.3\times10^7\ \mathrm{W/m^2} / 5.67\times10^{-8} \frac{W}{m^2 K^4}\right)^{1/4} = 5800\ K \qquad (\text{actual} \sim 6000 K)$$

What is T_{earth} if earth is a blackbody?

$$P_{hitting\ earth} = P_{radiated} \Rightarrow 1370 \frac{W}{m^2} \pi R_{earth}^2 = \sigma T^4 4\pi R_{earth}^2$$

$$T_{earth} = \left(1370\ \mathrm{W/m^2}/4 \ / 5.67\times10^{-8} \frac{W}{m^2K^4}\right)^{1/4} = 280\ K$$

Actual $T_{earth} \sim 288\ K$

Book says $\sim 30\%$ of sunlight is reflected $\Rightarrow T = 280K \cdot 0.7^{1/4} = 256\ K$

But this gets complicated because changes the amount radiated.

Greenhouse gases leave $P_{hitting\ earth} \sim$ unchanged but

$$P_{radiated} = e\, \sigma T^4\, 4\pi R_{earth}^2$$

e decreases because of changed wavelength

Wavelength from sun $\sim 20\times$ smaller than radiated from earth

Section 7.5 Debye Theory of Solids

This section gives a better treatment of vibrations in solids (phonons) which leads to more realistic behavior as $T \to$ small

The Einstein model of vibrations in solid had every atom oscillating at the same frequency.

Prob 3.25 $\to$ $C_V = 3Nk_B \dfrac{(\varepsilon/kT)^2 e^{\varepsilon/kT}}{(e^{\varepsilon/kT}-1)^2}$ $\quad N = \#$ of atoms $\quad \varepsilon = hf$ ← frequency

Problem: Atoms don't oscillate independently.

Next level of approximation assumes identical atoms coupled together by springs. Now, there are normal modes

etc

These are waves. Simplification to make derivation easier: there are 3 polarizations, they all have speed c_s, the speed is independent of f.

Use analogy Sound waves $\xleftrightarrow{\text{analogy}}$ light waves

phonons photons

Each mode has frequency $f = C_s/\lambda \Rightarrow \varepsilon = hf = \frac{hC_s}{\lambda}$

If you have a cube, the modes can be indexed by n_x, n_y, n_z

$\sin\left(\frac{\pi n_x}{L}x\right)\sin\left(\frac{\pi n_y}{L}y\right)\sin\left(\frac{\pi n_z}{L}z\right)$

$f = \frac{C_s}{L}\sqrt{n_x^2+n_y^2+n_z^2}$

As with light, the average n of a mode is for a harmonic oscillator

$\bar{n}_{pl} = \frac{1}{e^{\beta\varepsilon}-1} \qquad \varepsilon = \frac{hc_s}{2L}\sqrt{n_x^2+n_y^2+n_z^2}$

Repeat photon derivation

$$U = \underbrace{3}_{\text{polarization}} \sum_{n_x n_y n_z} \frac{hc_s}{2L}(n_x^2+n_y^2+n_z^2)^{1/2} \frac{1}{e^{\beta\frac{hc_s}{2L}\sqrt{n_x^2+n_y^2+n_z^2}}-1}$$

The big difference from photons is there is a max n_x, n_y, n_z for vibrations because the total number of modes $= 3\underbrace{N}_{\#\text{ of atoms}}$

$\Rightarrow 3\frac{1}{8}4\pi\int_0^{n_{max}} n^2\,dn = \frac{\pi}{2}n_{max}^3 \Rightarrow n_{max} = (6N/\pi)^{1/3}$

Convert the U sum to an integral

$U = 3\frac{1}{8}4\pi\frac{hc_s}{2L}\int_0^{n_{max}} n\frac{1}{e^{\beta\frac{hc_s}{2L}n}-1}n^2\,dn$ change variables $x = \frac{hc_s}{2L}\beta n$

$= \frac{3\pi}{2}\left(\frac{2L}{hc_s}\right)^3(kT)^4\int_0^{x_{max}} x^3/(e^x-1)\,dx \qquad x_{max} = \frac{hc_s}{2L}\beta n_{max}$ $\quad T$ is lurking in x

It's common to define the Debye temperature $T_D = \frac{hc_s n_{max}}{2k_B L} = \frac{hc_s}{2k_B}\left(\frac{6}{\pi}\frac{N}{V}\right)^{1/3}$

$x_{max} = T_D/T$ and

$\frac{3\pi}{2}k_B T^4\left(\frac{2k_B L}{hc_s}\right)^3 = 9Nk_B\frac{T^4}{T_D^3}$ Simplifies U

$$U = 9Nk_B\frac{T^4}{T_D^3}\int_0^{T_D/T}\frac{x^3}{e^x-1}\,dx$$

$T_D \simeq 90\,K$ for lead (heavy nuclei and weak springs)

$T_D \simeq 1900\,K$ for diamond (light nuclei and strong springs)

If $T \gg T_D \Rightarrow x \ll 1$ in integral $\frac{x^3}{e^x-1} \simeq x^2$

$U = 9Nk_B\frac{T^4}{T_D^3}\int_0^{T_D/T}x^2\,dx = 9Nk_B\frac{T^4}{T_D^3}\frac{1}{3}\frac{T_D^3}{T^3} = 3NkT$ ← equipartition!!

If $T \ll T_D$, $x_{max} \gg 1$ so can replace $T_D/T \to \infty$

$U = 9Nk_B\frac{T^4}{T_D^3}\int_0^\infty\frac{x^3}{e^x-1}\,dx = \frac{9\pi^4}{15}Nk_B T^4/T_D^3$

This result is important because the Einstein model goes to 0 too fast as $T \to 0$

The Einstein model has $C_V \sim \frac{1}{T^2} e^{-\varepsilon/kT}$

The Debye model has gentler dependence

$T \ll T_D \qquad C_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{12}{5}\pi^4 N k_B (T/T_D)^3$

The total heat capacity has contribution from electrons and phonons

Eq (7.68) $\quad U = \frac{3}{5} N \varepsilon_F + \frac{\pi^2}{4} N k_B^2 T^2/\varepsilon_F$

$\Rightarrow \quad C_V = \frac{\pi^2}{2} N k_B^2 T/\varepsilon_F$

$$\frac{C_V^{metal}}{N k_B} = \frac{\pi^2}{2}\frac{k_B T}{\varepsilon_F} + \frac{12}{5}\pi^4 (T/T_D)^3 \qquad \text{as } T \to 0$$

See Fig 7.28 (pg 311) for plot of C/T

Prob 7.57 Derive C_V for any T/T_D. Easier if you start with the original expression for U

$$U = \frac{3\pi}{2}\frac{hc_s}{2L}\int_0^{n_{max}} \frac{n^3}{e^{\beta h c_s n/2L} - 1}\, dn$$

$$\left(\frac{\partial U}{\partial T}\right)_V = \frac{3\pi}{2}\frac{hc_s}{2L}\int_0^{n_{max}} -\frac{n^3}{(e^{\frac{\beta h c_s n}{2L}}-1)^2}\, e^{\frac{\beta h c_s n}{2L}}\, \frac{h c_s n}{2L}\left(-\frac{\beta}{T}\right) dn \qquad x = \beta\frac{h c_s n}{2L}$$

$$= 9 N k_B \frac{T^3}{T_D^3}\int_0^{T_D/T} \frac{x^4 e^x}{(e^x - 1)^2}\, dx$$

Prob 7.59 Student answer

Prob 7.61 For liquid $C_V/Nk_B = \frac{1}{3}\frac{12}{5}\pi^4 (T/T_D)^3$

$c_s = 238\ m/s \qquad \rho = 0.145\ g/cm^3$

$$T_D = \frac{hc_s}{2k_B}\left(\frac{6}{\pi}\frac{N}{V}\right)^{1/3} \qquad \frac{N}{V} = \frac{\rho}{M_{He}} = \frac{0.145\times10^{-3}\,kg}{10^{-6}\,m^3 \ 4\cdot1.67\times10^{-27}\,kg} = 2.182\times10^{28}\,\frac{1}{m^3}$$

$$= \frac{6.626\times10^{-34}\,Js\ 238\,m/s}{2\ \ 1.38\times10^{-23}\,J/K}\left(\frac{6}{\pi}2.182\times10^{28}\frac{1}{m^3}\right)^{1/3} = 19.8\,K$$

$$C_V/Nk_B = \left(\frac{4}{5}\frac{\pi^4}{(19.8\,K)^3}\right)T^3 = 0.0100\,K^{-3}\,T^3$$

measured $\frac{C_V}{Nk_B} = \left(\frac{1}{4.67\,K}\right)^3 T^3 = 0.0098\,K^{-3}\,T^3$

great agreement