Chapter QIS - Quantum Information Science

In the remaining lectures, I will be treating topics affecting the "2nd Quantum Revolution". Because most of this material is <u>not</u> in the text book, I will number the important equations in these notes. Important equations will be in red.

20th Century
1st Quantum Revolution: Use QM to understand nature: fundamental particles, nuclear, atomic, chemistry, lasers, ...

21st Century
2nd Quantum Revolution: Manipulation of quantum phenomena: actively create, control, and probe quantum states, quan. sensing, quan. computing, quan. communication, quan. simulation, info, ...

Motivate why 2 state systems are interesting (Chap QIS slides 1-5)
The possible importance of quantum computing → focus on quantum systems that only have 2 effective states.
Examples: spin ½, not harmonic oscillator — never in
⇉ connect

One of the states will be "0" and the other will be "1".
Example: spin ½ 0 is $m = 1/2$ 1 is $m = -1/2$

New notation: column and Dirac bra-ket

(1) $\psi(x) = a_0 \psi_0(x) + a_1 \psi_1(x)$

(2) $|\psi\rangle \longleftrightarrow \begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

$$\int \psi^*(x)\psi(x)\,dx = \int [a_0^* \psi_0^*(x) + a_1^* \psi_1^*(x)][a_0\psi_0(x) + a_1\psi_1(x)]\,dx$$

$$= |a_0|^2 \int \psi_0^*(x)\psi_0(x)\,dx + a_0^* a_1 \underbrace{\cancel{\int \psi_0^*(x)\psi_1(x)\,dx}}_{\to 0}$$
$$+ a_1^* a_0 \underbrace{\cancel{\int \psi_1^*(x)\psi_0(x)\,dx}}_{\to 0} + |a_1|^2 \int \psi_1^*(x)\psi_1(x)\,dx$$

(3) $\int \psi^*(x)\psi(x)\,dx = a_0^* a_0 + a_1^* a_1 = (a_0^* \quad a_1^*)\begin{pmatrix} a_0 \\ a_1 \end{pmatrix}$

(4) $\int \psi^*(x)\,\psi(x)\,dx = \langle\psi|\psi\rangle$

(5) $\phi(x) = b_0\psi_0(x) + b_1\psi_1(x)$

(6) $\int \phi^*(x)\,\psi(x)\,dx = \langle\phi|\psi\rangle = \begin{pmatrix} b_0^* & b_1^* \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = b_0^* a_0 + b_1^* a_1$

Matrix representation of operator An operator takes a state into another state

(7) $Q_{op}\,\psi(x) = \phi(x)$

(8) $\underline{\underline{Q}}\,|\psi\rangle = |\phi\rangle$

(9) $\begin{pmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix}$

(10) $b_0 = Q_{00}a_0 + Q_{01}a_1 \qquad b_1 = Q_{10}a_0 + Q_{11}a_1$

Any physical operator can be written as the sum of 4 matrices with real coefficients

(11) $\underline{\underline{1}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \underline{\underline{\sigma_x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(12) $\underline{\underline{Q}} = g_0\,\underline{\underline{1}} + g_x\,\underline{\underline{\sigma_x}} + g_y\,\underline{\underline{\sigma_y}} + g_z\,\underline{\underline{\sigma_z}}$

(13) $\begin{pmatrix} Q_{00} & Q_{01} \\ Q_{10} & Q_{11} \end{pmatrix} = \begin{pmatrix} g_0 + g_z & g_x - i g_y \\ g_x + i g_y & g_0 - g_z \end{pmatrix}$

(14) $g_0 = \dfrac{Q_{00} + Q_{11}}{2} \qquad g_z = \dfrac{Q_{00} - Q_{11}}{2} \qquad g_x = \dfrac{Q_{01} + Q_{10}}{2} \qquad g_y = \dfrac{Q_{10} - Q_{01}}{2i}$

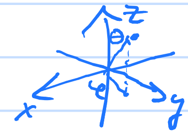
The σ_i are the Pauli matrices

What is being used for Qbits? (Chap QIS slide 6)

How to interpret $|\psi\rangle$?

(15) $|\psi\rangle = \cos\frac{\theta}{2}\,|0\rangle + \sin\frac{\theta}{2}\, e^{i\varphi}\,|1\rangle = \begin{pmatrix} \cos\theta/2 \\ \sin\theta/2\, e^{i\varphi} \end{pmatrix}$

point on unit sphere at θ, φ



Why? $\underline{\underline{Q}} = \sin\theta\cos\varphi\ \underline{\underline{\sigma_x}} + \sin\theta\sin\varphi\ \underline{\underline{\sigma_y}} + \cos\theta\ \underline{\underline{\sigma_z}}$

$$\underline{\underline{Q}} = \begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\varphi} \\ \sin\theta\, e^{i\varphi} & -\cos\theta \end{pmatrix}$$

$$\underline{\underline{Q}}\,|\psi\rangle = \begin{pmatrix} \cos\theta\cos\frac{\theta}{2} + \sin\theta\sin\frac{\theta}{2} \\ e^{i\varphi}\left(\sin\theta\cos\frac{\theta}{2} - \cos\theta\sin\frac{\theta}{2}\right) \end{pmatrix} = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi}\sin\theta/2 \end{pmatrix} = |\psi\rangle$$

$|\psi\rangle$ is an eigenstate of <u>this</u> $\underline{\underline{Q}}$ with eigenvalue 1.

The state with eigenvalue -1 is $|\psi_-\rangle = \begin{pmatrix} -e^{-i\varphi}\sin\theta/2 \\ \cos\theta/2 \end{pmatrix}$

Suppose there is a state characterized by θ', φ'. What is the probability that when you measure $\underline{\underline{Q}}$ you get 1?

(16) $P_{+1} = |\langle\psi|\psi'\rangle|^2 = \left| \left(\cos\frac{\theta}{2} \quad e^{-i\varphi}\sin\frac{\theta}{2}\right) \begin{pmatrix} \cos\theta'/2 \\ e^{i\varphi'}\sin\frac{\theta'}{2} \end{pmatrix} \right|$

$= \left\{\cos\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta'}{2}\right) + e^{i(\varphi'-\varphi)}\sin\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta'}{2}\right)\right] \left[\qquad \right]^*$

$= \cos^2\left(\frac{\alpha}{2}\right)$ $\qquad \alpha$ = angle between θ, φ and θ', φ'

Probability to get -1?

(17) $P_{-1} = 1 - P_+ = \sin^2\left(\frac{\alpha}{2}\right)$

Now do 2 Qbits. Choose an ordering of the states

$$(18) \quad \overset{0}{|0\rangle_1|0\rangle_2}, \quad \overset{1}{|1\rangle_1|0\rangle_2}, \quad \overset{2}{|0\rangle_1|1\rangle_2}, \quad \overset{3}{|1\rangle_1|1\rangle_2} \quad \leftarrow \text{binary}$$

$$(19) \quad |\psi\rangle = a_{00}|0\rangle_1|0\rangle_2 + a_{10}|1\rangle_1|0\rangle_2 + a_{01}|0\rangle_1|1\rangle_2 + a_{11}|1\rangle_1|1\rangle_2$$

$$\simeq \begin{pmatrix} a_{00} \\ a_{10} \\ a_{01} \\ a_{11} \end{pmatrix}$$

Can extend to N Qbits. There are 2^N coefficients in general. For separable states there are 2N coeff.

$$(20) \quad \text{separable} \quad |\psi\rangle = \left(\cos\tfrac{\theta_1}{2}|0\rangle_1 + e^{i\varphi_1}\sin\tfrac{\theta_1}{2}|1\rangle_1\right)\left(\cos\tfrac{\theta_2}{2}|0\rangle_2 + e^{i\varphi_2}\sin\tfrac{\theta_2}{2}|1\rangle_2\right)$$

etc

States that can <u>not</u> be written like Eq (20) mean some (all?) Qbits are <u>entangled</u>.
States with all Qbits entangled require exponential classical resources to simulate.

Qbits that are not entangled: the value of measurement on 1 Qbit does <u>not</u> affect the measurement on the other

Not entangled: $|\psi\rangle = |0\rangle_1|1\rangle_2$ $\qquad P_{ij}$ = prob. to measure i on 1, j on 2

$P_{00} = 0 \qquad P_{01} = 1 \qquad P_{10} = 0 \qquad P_{11} = 0$

Prob to measure 0 for 1 $= P_{00} + P_{01} = 1$

" " " 1 " " $= P_{10} + P_{11} = 0$

Not entangled $\quad |\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 + |1\rangle_1)\left(\frac{1}{\sqrt{3}}|0\rangle_2 - \sqrt{\frac{2}{3}}|1\rangle_2\right) = \frac{1}{\sqrt{6}}\left(|0\rangle_1|0\rangle_2 - \sqrt{2}|0\rangle_1|1\rangle_2 + |1\rangle_1|0\rangle_2 - \sqrt{2}|1\rangle_1|1\rangle_2\right)$

$P_{00} = 1/6, \quad P_{01} = 1/3, \quad P_{10} = 1/6, \quad P_{11} = 1/3$

If Qbit 2 = 1, prob Qbit 1 = 0 ? $\qquad P_{01}/(P_{01} + P_{11}) = 1/2$

" " " = 0, " " " ? $\qquad P_{00}/(P_{00} + P_{10}) = 1/2$

Entangled $\quad |\psi\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_1|0\rangle_2 - |1\rangle_1|1\rangle_2\right)$

$P_{00} = 1/2 \qquad P_{01} = 0 \qquad P_{10} = 0 \qquad P_{11} = 1/2$

If Qbit 2=1, prob Qbit 1 = 0 ? $\qquad P_{01}/(P_{01} + P_{11}) = 0$

" 0 " " $\qquad P_{00}/(P_{00} + P_{10}) = 1$

Entanglement is at the heart of the weirdest quantum effects. In 1935, Einstein, Podolsky, and Rosen (EPR) described a thought experiment with entanglement which "showed" QM is not complete. Bohm gave a variant in 1951. One suggestion was "hidden variables". Bell showed in 1964 that QM violates <u>all</u> possible local hidden variable theories.

Basic idea of hidden variable: There is some quantity that we haven't figured out how to measure that affects the measurement we are interested in.

Example: Suppose you prepare a Qbit (where quantum mechanically we would say $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$). It is actually $|\psi(\lambda)\rangle$ where λ is "hidden". λ = determines what happens

Do a measurement with $Q = \sigma_z$? $\frac{1}{2}$ of the starting λ give $+1$, $\frac{1}{2}$ give -1

" " " " $Q = \sigma_x$? 100% " " λ " $+1$

etc

Bohm variation of EPR: (1) Start 2 Qbits close together in an entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2)$. (2) Separate to large distance (send Qbit 1 left to Alice and Qbit 2 right to Bob). (3) They measure $Q = \sigma_z$. (4) $\frac{1}{2}$ the time they measure $+1$ and $\frac{1}{2}$ the time they measure -1. (5) <u>BUT</u> if Alice measures $+1$, Bob <u>always</u> measure -1 and vice versa. (6) How did this happen? Hidden variables! When the Qbits were together, their hidden variables were set so if Alice measures $+1$ then Bob measures -1 and vice versa.

Bell showed this idea could <u>never</u> explain a more complicated version.

Bell variation: (1) Alice will randomly measure z or x direction. Bob will measure randomly in the $-\frac{1}{\sqrt{2}}(\hat{x}+\hat{z})$ or $\frac{1}{\sqrt{2}}(\hat{x}-\hat{z})$ direction.

(2) Number their 2 possible measurements A_1, A_2 and B_1, B_2

(3) The value they measure will be a_1 or a_2 and b_1 or b_2

(4) V will be the joint value $4a_1b_1$ or $4a_1b_2$ or $4a_2b_1$ or $-4a_2b_2$ depending on the pair of measurements done.

(5) Average V over a bunch of runs $V/N = \frac{\sum a_1b_1}{N/4} + \frac{\sum a_1b_2}{N/4} + \frac{\sum a_2b_1}{N/4} - \frac{\sum a_2b_2}{N/4}$

(21) $\langle V\rangle = \langle a_1b_1\rangle + \langle a_1b_2\rangle + \langle a_2b_1\rangle - \langle a_2b_2\rangle$

We can find the limits of the hidden variables world.

$\bar{V} = a_1 b_1 + a_1 b_2 + a_2 b_1 - a_2 b_2 = (a_1 + a_2) b_1 + (a_1 - a_2) b_2$

Look at examples:

Hidden variables =>	$a_1 = +1$	$a_2 = +1$	$b_1 = -1$	$b_2 = -1$	$\bar{V} = -2$
	$+1$	-1	$+1$	$+1$	$\bar{V} = 2$
	$+1$	-1	-1	-1	$\bar{V} = -2$

etc

by logic if $a_1 + a_2 = \pm 2 \quad a_1 - a_2 = 0 \Rightarrow \bar{V} = \pm 2$

$a_1 + a_2 = 0 \quad a_1 - a_2 = \pm 2 \Rightarrow \bar{V} = \pm 2$

(22) Hidden variables $\Rightarrow -2 \le \langle V \rangle \le 2$

What is the quantum result for this state? We need to calculate the quantum version of the different averages

(23) $\langle Q \rangle = \langle \psi | Q | \psi \rangle$

Below I show for $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2)$

(24) $\langle \psi | (\hat{d}_1 \cdot \vec{\sigma}_1)(\hat{d}_2 \cdot \vec{\sigma}_2) | \psi \rangle = -\hat{d}_1 \cdot \hat{d}_2$

(25) Quantum $\Rightarrow V = -\hat{z} \cdot \left[-\frac{1}{\sqrt{2}}(\hat{x}+\hat{z})\right] - \hat{z} \cdot \left[\frac{1}{\sqrt{2}}(\hat{x}-\hat{z})\right] - \hat{x} \cdot \left[-\frac{1}{\sqrt{2}}(\hat{x}+\hat{z})\right] - -\hat{x} \cdot \left[\frac{1}{\sqrt{2}}(\hat{x}-\hat{z})\right]$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 2\sqrt{2}$

This is larger than the range allowed by local hidden variable theory. Experiments match the quantum result!

Show $\langle \hat{Q} \rangle = \langle \psi | (\vec{d}_1 \cdot \vec{\sigma}_1)(\vec{d}_2 \cdot \vec{\sigma}_2) | \psi \rangle = -\hat{d}_1 \cdot \hat{d}_2$ for $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_1 |1\rangle_2 - |1\rangle_1 |0\rangle_2)$

$$\hat{d}_1 \cdot \vec{\sigma} = \begin{pmatrix} d_{1z} & d_{1x} - i d_{1y} \\ d_{1x} + i d_{1y} & -d_{1z} \end{pmatrix}$$

$$\langle \hat{Q} \rangle = \tfrac{1}{2} \langle 0 | \hat{d}_1 \cdot \vec{\sigma} | 0 \rangle \langle 1 | \hat{d}_2 \cdot \vec{\sigma} | 1 \rangle + \tfrac{1}{2} \langle 1 | \hat{d}_1 \cdot \vec{\sigma} | 1 \rangle \langle 0 | \hat{d}_2 \cdot \vec{\sigma} | 0 \rangle - \tfrac{1}{2} \langle 0 | \hat{d}_1 \cdot \vec{\sigma} | 1 \rangle \langle 1 | \hat{d}_2 \cdot \vec{\sigma} | 0 \rangle$$
$$- \tfrac{1}{2} \langle 1 | \hat{d}_1 \cdot \vec{\sigma} | 0 \rangle \langle 0 | \hat{d}_2 \cdot \vec{\sigma} | 1 \rangle$$
$$= \tfrac{1}{2} d_{1z}(-d_{2z}) + \tfrac{1}{2}(-d_{1z}) d_{2z} \underbrace{- \tfrac{1}{2}(d_{1x} - i d_{1y})(d_{2x} + i d_{2y}) - \tfrac{1}{2}(d_{1x} + i d_{1y})(d_{2x} - i d_{2y})}_{-\frac{1}{2}[d_{1x} d_{2x} - i d_{1y} d_{2x} + i d_{1x} d_{2y} + d_{1y} d_{2y} + d_{1x} d_{2x} + i d_{1y} d_{2y} \cdots}$$
$$= -d_{1z} d_{2z} - d_{1x} d_{2x} - d_{1y} d_{2y}$$
$$= -\vec{d}_1 \cdot \vec{d}_2$$

Quantum communication is one of the near term "products". To get quantitative, I need to introduce some ideas from quantum computing. One concept is "gates": take input quantum state into a new state

(26) $G|\psi\rangle = |\phi\rangle$

Quantum gates are unitary

(27) $G|\psi_1\rangle = |\phi_1\rangle \qquad G|\psi_2\rangle = |\phi_2\rangle \qquad \langle\phi_1|\phi_2\rangle = \langle\psi_1|\psi_2\rangle$

Hadamard Gate acts on only 1 Qbit

(28) $|\psi\rangle_{in} — \boxed{H} —_{out} |\phi\rangle$

(29) $|0\rangle — \boxed{H} — \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \qquad |1\rangle — \boxed{H} — \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

What happens if the Hadamard gate 2 times?

$|0\rangle — \boxed{H} — \boxed{H} — ? \qquad |0\rangle \to \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \to \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\right] = |0\rangle$

$|1\rangle — \boxed{H} — \boxed{H} — |1\rangle$

Not (or Pauli X) Gate acts on 1 Qbit

(30) $|0\rangle — \oplus — |1\rangle \qquad |1\rangle — \oplus — |0\rangle$

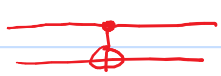
This gate swaps the 0 and 1 character

Phase shift Gate acts on 1 Qbit

(31) $|0\rangle — \boxed{\varphi} — |0\rangle \qquad |1\rangle — \boxed{\varphi} — e^{i\varphi}|1\rangle$

This gate gives a phase shift between the $|0\rangle$ and $|1\rangle$

Controlled Not (CNOT or CX) gate acts on 2 Qbits

(32) 

(33) $|0\rangle_1|0\rangle_2 \to |0\rangle_1|0\rangle_2 \qquad |0\rangle_1|1\rangle_2 \to |0\rangle_1|1\rangle_2$

$|1\rangle_1|0\rangle_2 \to |1\rangle_1|1\rangle_2 \qquad |1\rangle_1|1\rangle_2 \to |1\rangle_1|0\rangle_2$

Do "Not gate" on Qbit 2 if Qbit 1 is 1. Don't do anything to Qbit 2 if Qbit 1 is 0.

What do you get when you do a Hadamard, a $\varphi=\pi$ phase, and Hadamard gates?

$|0\rangle \xrightarrow{H} \frac{|0\rangle+|1\rangle}{\sqrt{2}} \xrightarrow{\varphi=\pi} \frac{|0\rangle-|1\rangle}{\sqrt{2}} \xrightarrow{H} \frac{1}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) - \frac{1}{\sqrt{2}}\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = |1\rangle$

$|1\rangle \xrightarrow{H} \frac{|0\rangle-|1\rangle}{\sqrt{2}} \xrightarrow{\varphi=\pi} \frac{|0\rangle+|1\rangle}{\sqrt{2}} \xrightarrow{H} \frac{1}{\sqrt{2}}\left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}\left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) = |0\rangle$

Not gate

There are many gates that have been defined (Quantum Logic Gate)
The gates aren't completely independent (some can be gotten from combination of others).
These gates have led to powerful quantum algorithms. Many of them are exponentially faster than the corresponding algorithms for classical computers. (ChapQIS slides 7-10)

Return to quantum communication.
To understand part of the excitement for quantum communication we need to understand the "No Cloning" theorem: It is impossible to make a copy of an unknown state without irrevocably changing it.

Proof for Qbit

Suppose there is a "Clone" operation

$$|\phi\rangle_1 |0\rangle_2 = (a_0 |0\rangle_1 + a_1 |1\rangle_1)|0\rangle_2$$

Want

$$|\phi\rangle_1 |0\rangle_2 \xrightarrow{\text{clone}} |\phi\rangle_1 |\phi\rangle_2 = (a_0 |0\rangle_1 + a_1 |1\rangle_1)(a_0 |0\rangle_2 + a_1 |1\rangle_2)$$

Is it possible? Look at the cases where $a_0 = 1, a_1 = 0$ or $a_0 = 0, a_1 = 1$

$$|0\rangle_1 |0\rangle_2 \xrightarrow{\text{clone}} |0\rangle_1 |0\rangle_2 \quad \text{and} \quad |1\rangle_1 |0\rangle_2 \xrightarrow{\text{clone}} |1\rangle_1 |1\rangle_2$$

This means

(34) $$|\phi_1\rangle |0\rangle_2 = a_0 |0\rangle_1 |0\rangle_2 + a_1 |1\rangle_1 |0\rangle_2 \xrightarrow{\text{clone}} \underbrace{a_0 |0\rangle_1 |0\rangle_2 + a_1 |1\rangle_1 |1\rangle_2 \neq |\phi_1\rangle |\phi_2\rangle}_{\text{You didn't clone! You entangled.}}$$

The "No Cloning" Theorem has huge implications for quantum Communication. There is no way to eavesdrop in an undetectable way. This is ruled out by fundamental QM! Any attempt to eavesdrop is detectable. (ChapQIS slides 11-19)

Another interesting Quantum communication application: Quantum teleportation. We know that there is no cloning, but if you "destroy" the initial state then you can have a far distant Qbit exactly go into that state. In principle 100%.

Steps:

1) Alice prepares 2 Qbits into the entangled state $|\psi\rangle = \frac{|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2}{\sqrt{2}}$

2) Alice sends Qbit 1 to Bob

3) Alice has a 3rd Qbit but doesn't know the state $|\phi\rangle = a_0|0\rangle_3 + a_1|1\rangle_3$

4) Alice does a C-Not on 2,3 with 2 the control and then Hadamard on 2

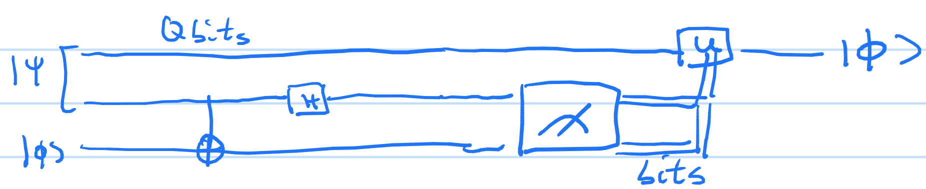
5) Alice then measures both bits getting $|0\rangle_2|0\rangle_3$, $|0\rangle_2|1\rangle_3$, $|1\rangle_2|0\rangle_3$, or $|1\rangle_2|1\rangle_3$

6) Alice sends to Bob the info on which state she measured.

7) Depending on what Alice measured Bob does a gate on his Qbit.

8) Bob's Qbit is now $a_0|0\rangle_1 + a_1|1\rangle_2$ state $|\phi\rangle$!!

(Chap QIS slides 20-23)



If the original Qbit 3 had been entangled with Qbit 4, then Qbit 1 would now be in exactly that entangled state with 4.

SHOW

use $|a\rangle_1|b\rangle_2|c\rangle_3 = |abc\rangle$

original $= \frac{|0\rangle_1|1\rangle_2 - |1\rangle_1|0\rangle_2}{\sqrt{2}}(a_0|0\rangle_3 + a_1|1\rangle_3) = \frac{a_0}{\sqrt{2}}|010\rangle - \frac{a_0}{\sqrt{2}}|100\rangle + \frac{a_1}{\sqrt{2}}|011\rangle - \frac{a_1}{\sqrt{2}}|101\rangle$

CNot original $= \frac{a_0}{\sqrt{2}}|011\rangle - \frac{a_0}{\sqrt{2}}|100\rangle + \frac{a_1}{\sqrt{2}}|010\rangle - \frac{a_1}{\sqrt{2}}|101\rangle$

$\boxed{H}_2$ $= \frac{a_0}{2}|001\rangle - \frac{a_0}{2}|011\rangle - \frac{a_0}{2}|100\rangle - \frac{a_0}{2}|110\rangle + \frac{a_1}{2}|000\rangle - \frac{a_1}{2}|010\rangle - \frac{a_1}{2}|101\rangle - \frac{a_1}{2}|111\rangle$

$= \frac{1}{2}\Big[(-a_0|1\rangle_1 + a_1|0\rangle_1)|00\rangle_{23} + (a_0|0\rangle_1 - a_1|1\rangle_1)|01\rangle_{23} - (a_0|1\rangle_1 + a_1|0\rangle_1)|10\rangle_{23} - (a_0|0\rangle_1 + a_1|1\rangle_1)|11\rangle_{23}\Big]$

If measure $|11\rangle_{23}$, then Bob has $a_0|0\rangle_1 + a_1|1\rangle_1 = |\phi\rangle$ ✓

" " $|10\rangle_{23}$, " " " $a_0|1\rangle_1 + a_0|0\rangle_1$. Do X-gate $a_0|0\rangle_1 + a_1|1\rangle_1 = |\phi\rangle$ ✓

" " $|01\rangle_{23}$, " " " $a_0|0\rangle_1 - a_1|1\rangle_1$. Do $\varphi = \pi$ gate $a_0|0\rangle_1 + a_1|1\rangle_1 = |\phi\rangle$ ✓

" " $|00\rangle$, " " " $a_0|1\rangle_1 - a_0|0\rangle_1$. Do X gate and $\varphi = \pi$ gate $a_0|0\rangle_1 + a_1|1\rangle_1 = |\phi\rangle$ ✓

Quantum sensing – Use some feature of a quantum system to accurately measure some property. Basic idea: find a quantum system particularly sensitive to that property (ChmpQIS slides 24-31)

Many other examples: Rydberg atoms to detect electric fields, microwaves, THz fields