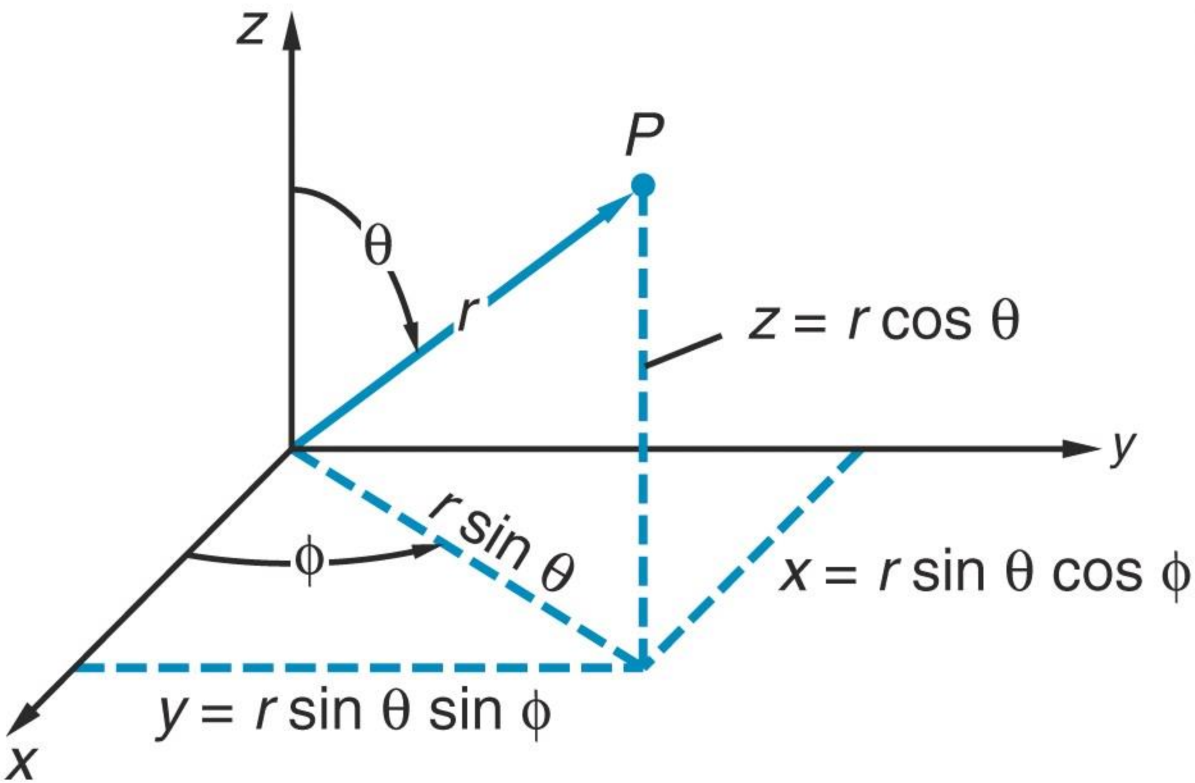
Spherical Coordinates



Range of variables

Cartesian

$x, y, z: -\infty \rightarrow +\infty$

Spherical

$r: 0 \rightarrow +\infty$

$\theta: 0 \rightarrow \pi$

$\phi: 0 \rightarrow 2\pi$

$$\frac{\partial^2 F}{\partial x^2}+\frac{\partial^2 F}{\partial y^2}+\frac{\partial^2 F}{\partial z^2}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial F}{\partial r}\right)+\frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial F}{\partial\theta}\right)+\frac{1}{r^2\sin^2(\theta)}\frac{\partial^2 F}{\partial\phi^2}$$

Angular Momentum, Classical

The definition of angular momentum is

$$\vec{L} = \vec{r} \times \vec{p} \qquad L_x = y\,p_z - z\,p_y \qquad L_y = z\,p_x - x\,p_z \qquad L_z = x\,p_y - y\,p_x$$

Properties of cross products?

Direction perpendicular to both vectors.

Right hand rule.

Magnitude is the product of magnitudes times sin of angle between

Cyclic nature & distributive property

$$\hat{i} = \hat{j} \times \hat{k} \qquad \hat{k} = \hat{i} \times \hat{j} \qquad \hat{j} = \hat{k} \times \hat{i}$$

$$\hat{i} = -\hat{k} \times \hat{j} \qquad \hat{k} = -\hat{j} \times \hat{i} \qquad \hat{j} = -i \times \hat{k}$$

Question #1

Classical: An object at position **r** has an angular momentum **L** = **r** X **p**. If **L** is constant which statement is correct?

(a) The force on the object must be 0.

(b) The force in the direction of **r** must be 0.

(c) The force perpendicular to the direction **r** must be 0.

(d) None of the above are correct.

Question #2

Classical: Which statement is true for an object at position $\mathbf{r}(t)$ with $\mathbf{L}$ constant? $|\mathbf{r}| = r = (x^2 + y^2 + z^2)^{1/2}$

(a) The magnitude of the force only depends on r.

(b) The potential energy, U, only depends on r.

(c) The potential energy perpendicular to r must be 0.

(d) The force must be a constant.

(e) The potential energy depends linearly on x or y or z.

Angular Momentum, Important

The eigenstate of one component is not an eigenstate of the other two components (except j = n = m = 0). **It is not possible to have a quantum system with defined L_x, L_y, and L_z at the same time.**

By convention L_z is almost always the special, well defined component of **L**.

m is an integer

In spherical coordinates [$x = r \sin(\theta) \cos(\phi)$, $y = r \sin(\theta) \sin(\phi)$] the eigenstate has simple form.

$(x + i\, y)/r = \sin(\theta)\, e^{i\,\phi}$ ⇨ $[(x + i\, y)/r]^m = \sin^m(\theta)\, e^{i\,m\,\phi}$

Total Angular Momentum, Quantum

The squared angular momentum arises from summing and squaring the components. In terms of spherical coordinates

($x = r\sin(\theta)\cos(\phi)$, $y = r\sin(\theta)\sin(\phi)$, $z = r\cos(\theta)$),

L^2, has a relatively simple form

$$L^2_{op} = -\hbar^2\left[\frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2(\theta)}\frac{\partial^2}{\partial\phi^2}\right]$$

The eigenstates are the spherical harmonics which are also eigenstates of L_z. The eigenvalues are $\ell(\ell+1)\hbar^2$

$$L^2_{op}\ Y_{\ell m}(\theta,\phi) = \hbar^2\ \ell(\ell+1)\ Y_{\ell m}(\theta,\phi)$$

$$L_z\ Y_{\ell m}(\theta,\phi) = \hbar m\ Y_{\ell m}(\theta,\phi)$$

Spherical Harmonics

TABLE 7-1 Spherical harmonics

$l = 0$	$m = 0$	$Y_{00} = \sqrt{\frac{1}{4\pi}}$
$l = 1$	$m = 1$	$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi}$
	$m = 0$	$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$
	$m = -1$	$Y_{1-1} = \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{-i\phi}$
$l = 2$	$m = 2$	$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{2i\phi}$
	$m = 1$	$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{i\phi}$
	$m = 0$	$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1)$
	$m = -1$	$Y_{2-1} = \sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{-i\phi}$
	$m = -2$	$Y_{2-2} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{-2i\phi}$

Note: A 3-D color representation of the spherical harmonics is on the Internet at http://ww3.uniovi.es/~quimica.fisica/qeg/harmonics/charmonics.html

Angular Momentum, Values

ℓ can be 0, 1, 2, 3, …

m can go from $-\ell$ to ℓ and is an integer

The number of oscillations in the ϕ coordinate is $|m|$.

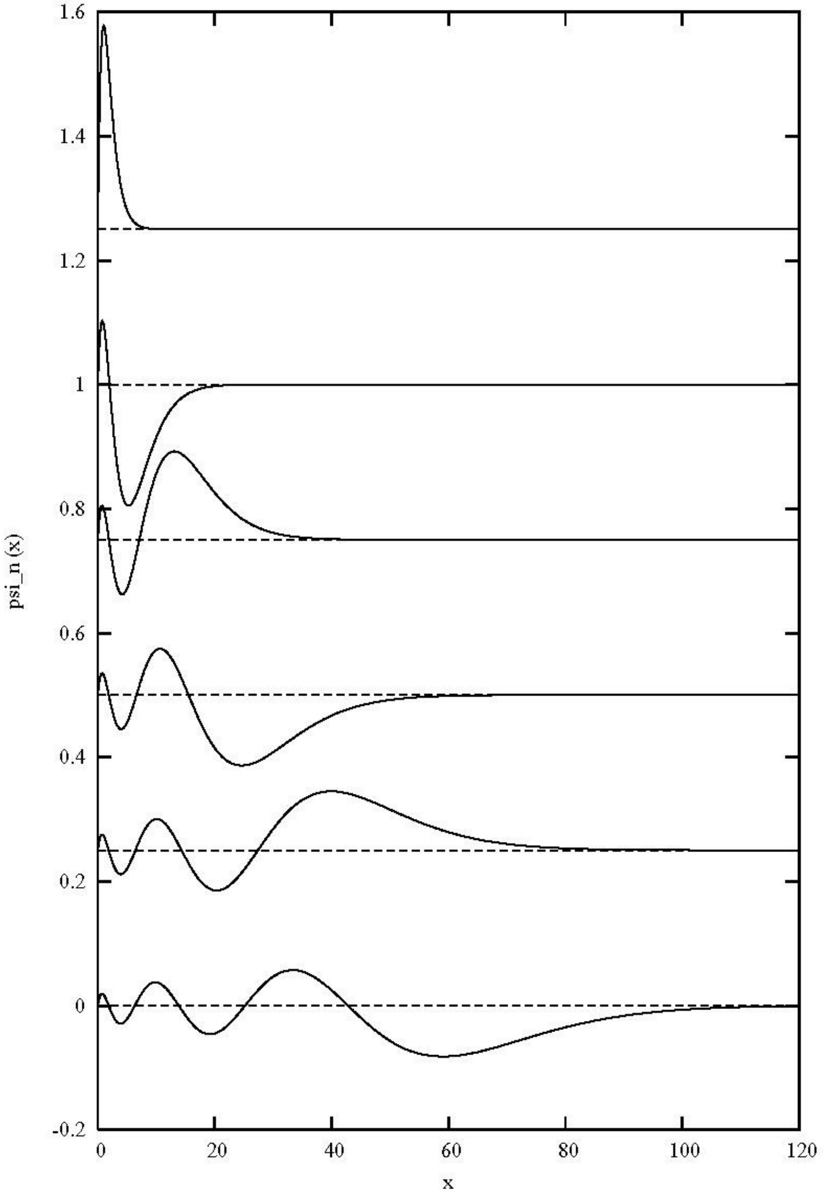
The number of oscillations in the θ coordinate is $\ell - |m|$

TABLE 7.1 Some Hydrogen Atom Wave Functions

n	l	m_l	$R(r)$	$\Theta(\theta)$	$\Phi(\phi)$
1	0	0	$\frac{2}{a_0^{3/2}}e^{-r/a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
2	0	0	$\frac{1}{(2a_0)^{3/2}}\left(2-\frac{r}{a_0}\right)e^{-r/2a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
2	1	0	$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$	$\sqrt{\frac{3}{2}}\cos\theta$	$\frac{1}{\sqrt{2\pi}}$
2	1	±1	$\frac{1}{\sqrt{3}(2a_0)^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$	$\mp\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$
3	0	0	$\frac{2}{(3a_0)^{3/2}}\left(1-\frac{2r}{3a_0}+\frac{2r^2}{27a_0^2}\right)e^{-r/3a_0}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2\pi}}$
3	1	0	$\frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0}-\frac{r^2}{6a_0^2}\right)e^{-r/3a_0}$	$\sqrt{\frac{3}{2}}\cos\theta$	$\frac{1}{\sqrt{2\pi}}$
3	1	±1	$\frac{8}{9\sqrt{2}(3a_0)^{3/2}}\left(\frac{r}{a_0}-\frac{r^2}{6a_0^2}\right)e^{-r/3a_0}$	$\mp\frac{\sqrt{3}}{2}\sin\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$
3	2	0	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\sqrt{\frac{5}{8}}(3\cos^2\theta-1)$	$\frac{1}{\sqrt{2\pi}}$
3	2	±1	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\mp\sqrt{\frac{15}{4}}\sin\theta\cos\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm i\phi}$
3	2	±2	$\frac{4}{27\sqrt{10}(3a_0)^{3/2}}\frac{r^2}{a_0^2}e^{-r/3a_0}$	$\frac{\sqrt{15}}{4}\sin^2\theta$	$\frac{1}{\sqrt{2\pi}}e^{\pm 2i\phi}$

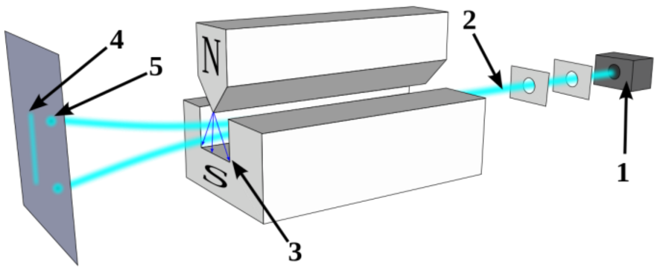
Coulomb Potential (-1/r), $\ell = 0$

r R(r)

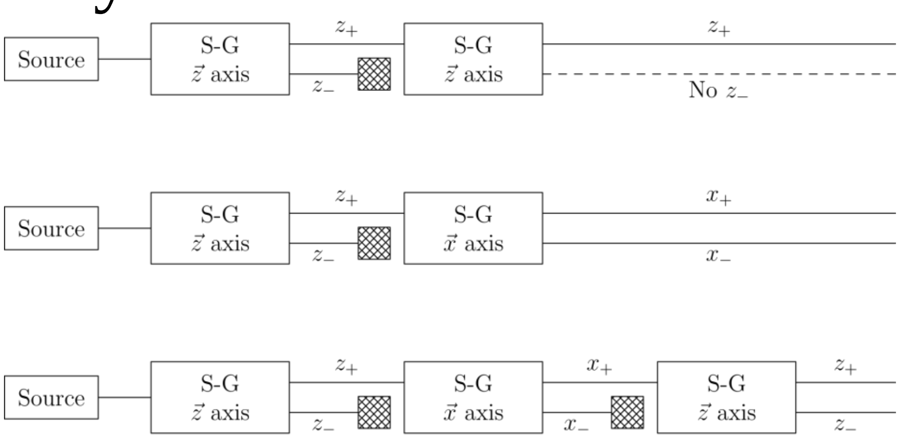


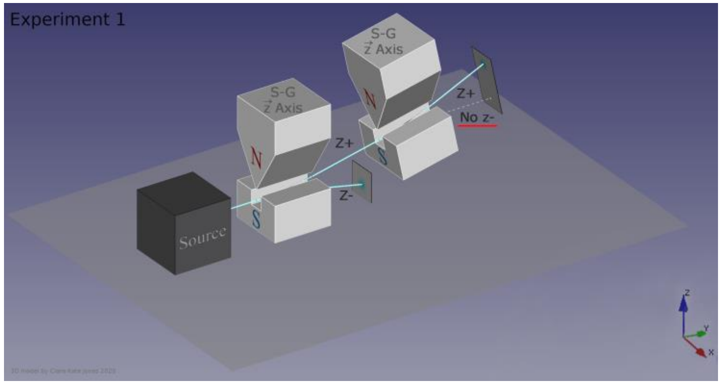
Stern-Gerlach Experiment

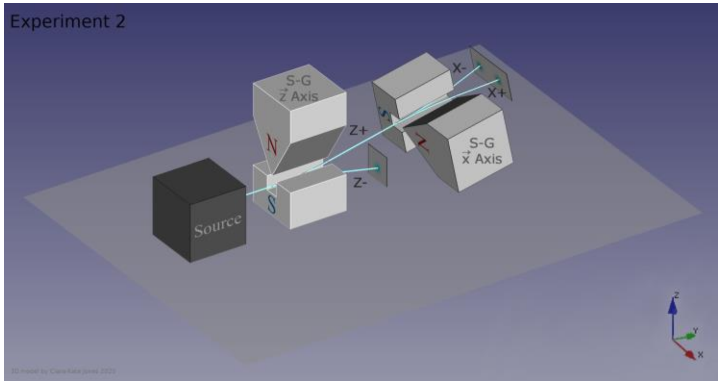
Basic for spin ½

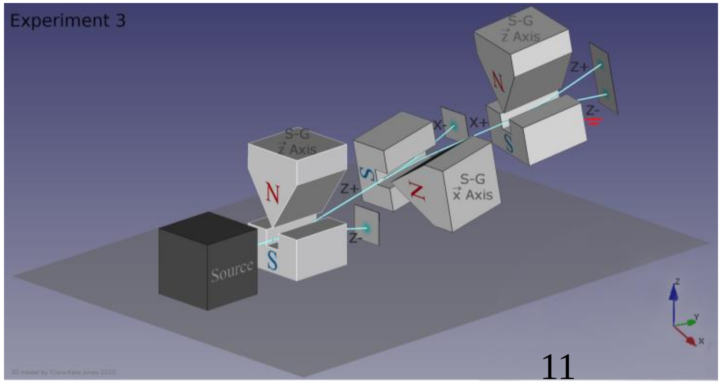


Can show the spin ½ for any direction





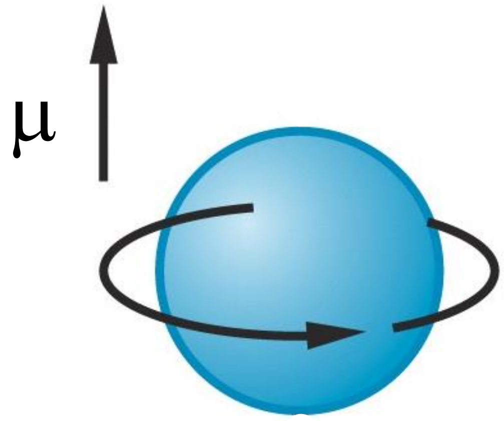




Trapping Neutral Particles

$$\mathrm{PE} = -\vec{\mu} \bullet \vec{\mathrm{B}}$$

$$\vec{\mu} \bullet \hat{\mathrm{B}} \cong \mathrm{constant}$$

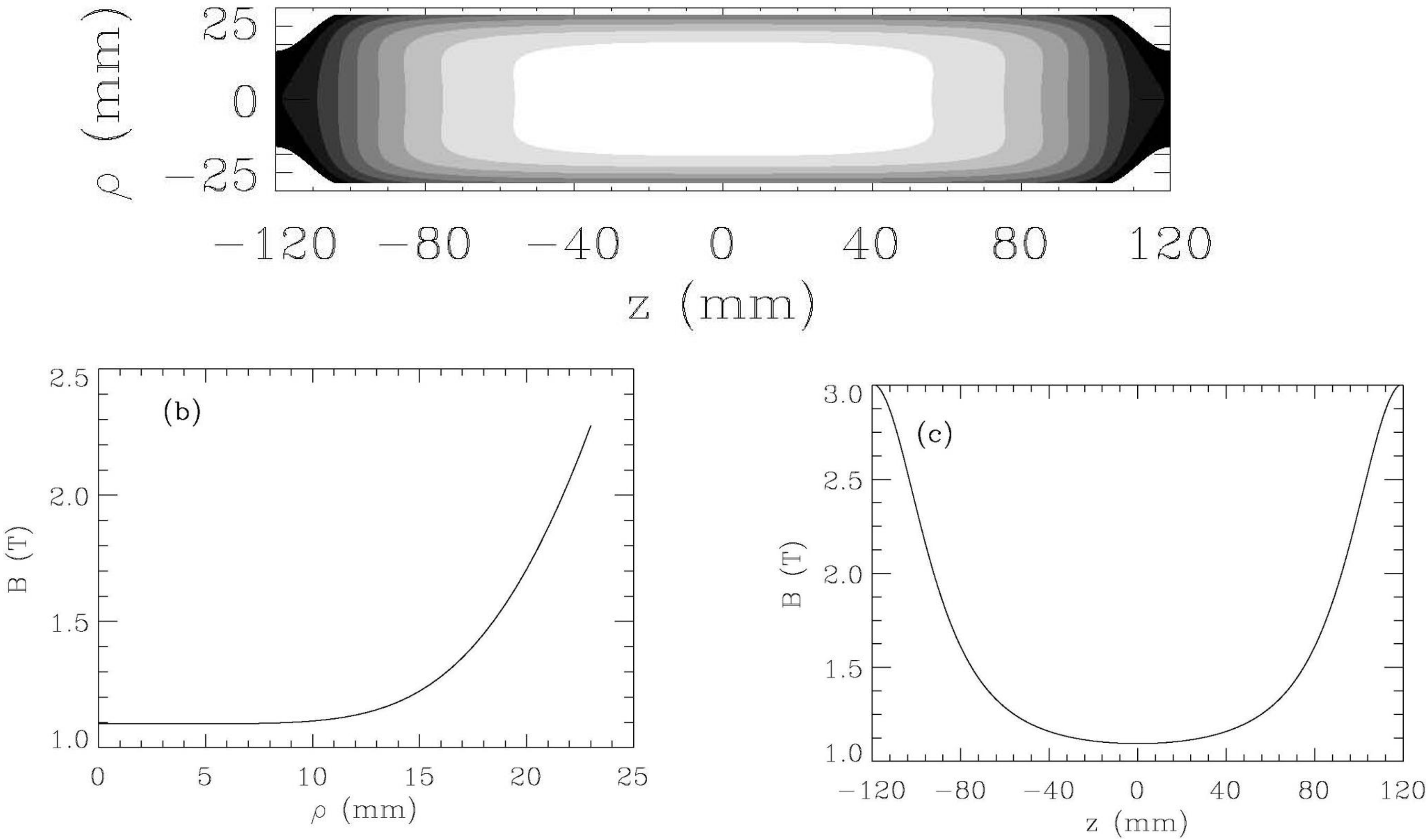


$$\mathrm{PE} = -\left(\vec{\mu} \bullet \hat{\mathrm{B}}\right) \mathrm{B}(x, y, z)$$

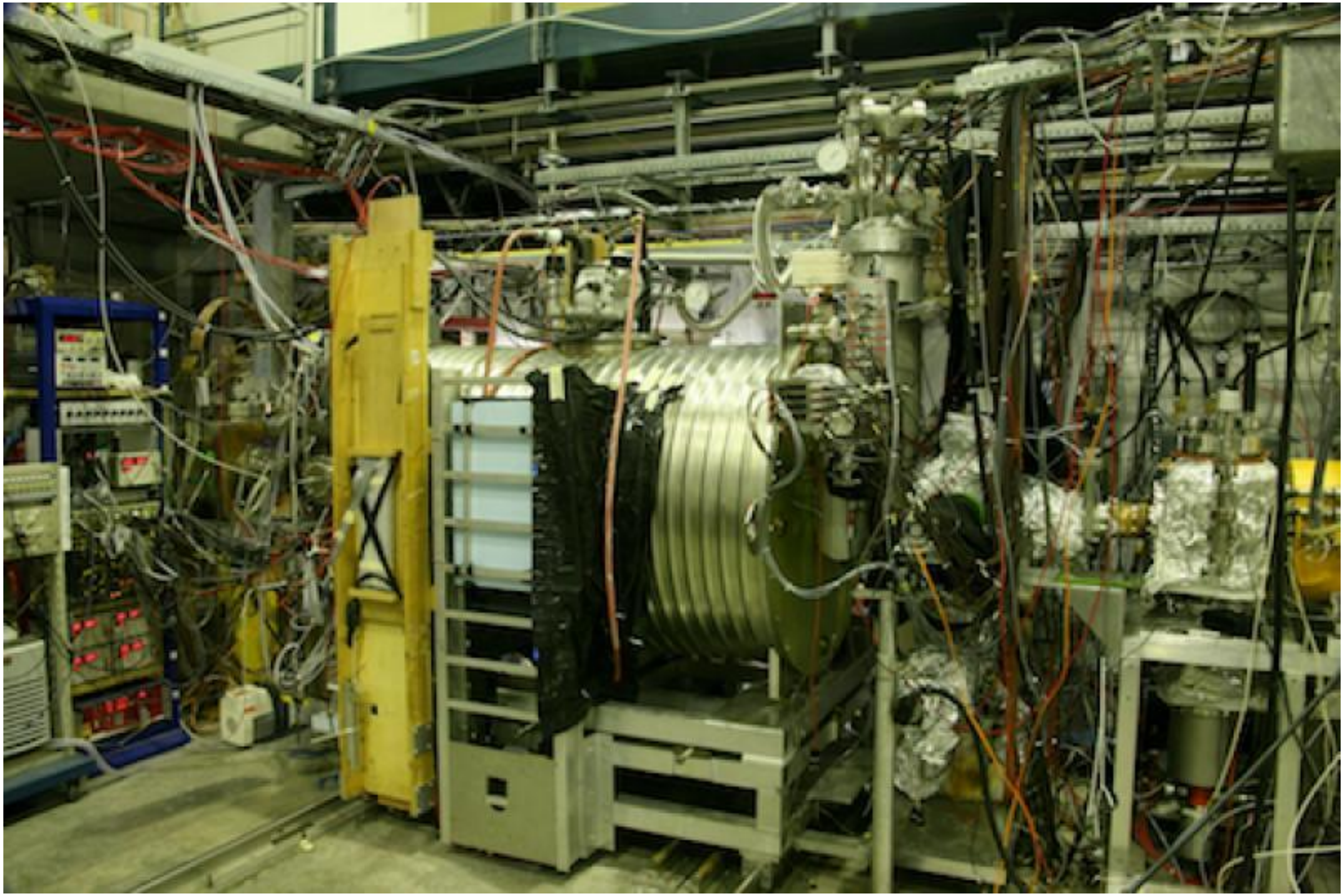
μ for ground state ~2/3 K/T

Actually 4 states: 2 trapped and 2 ejected

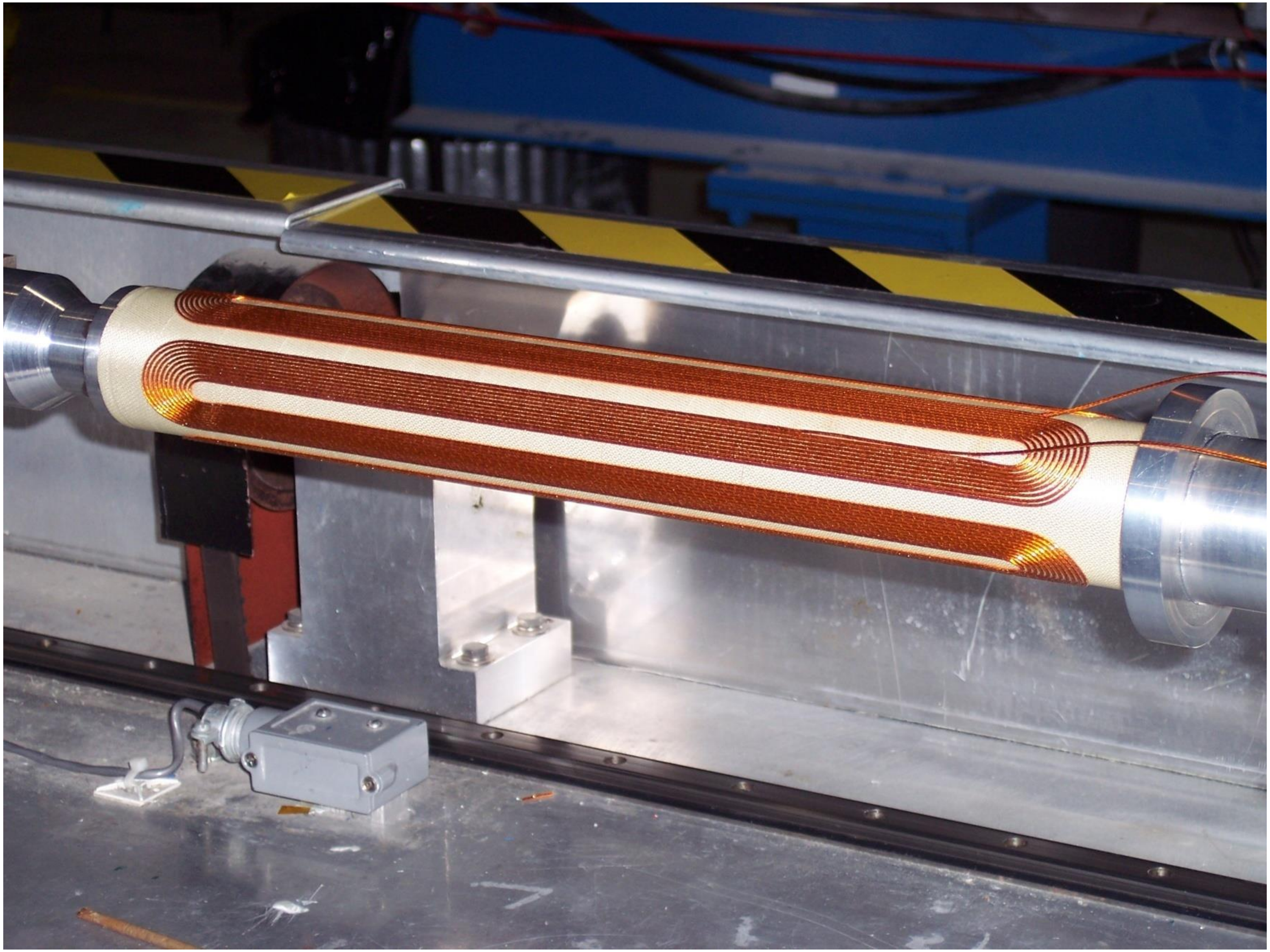
AntiHydrogen Trap



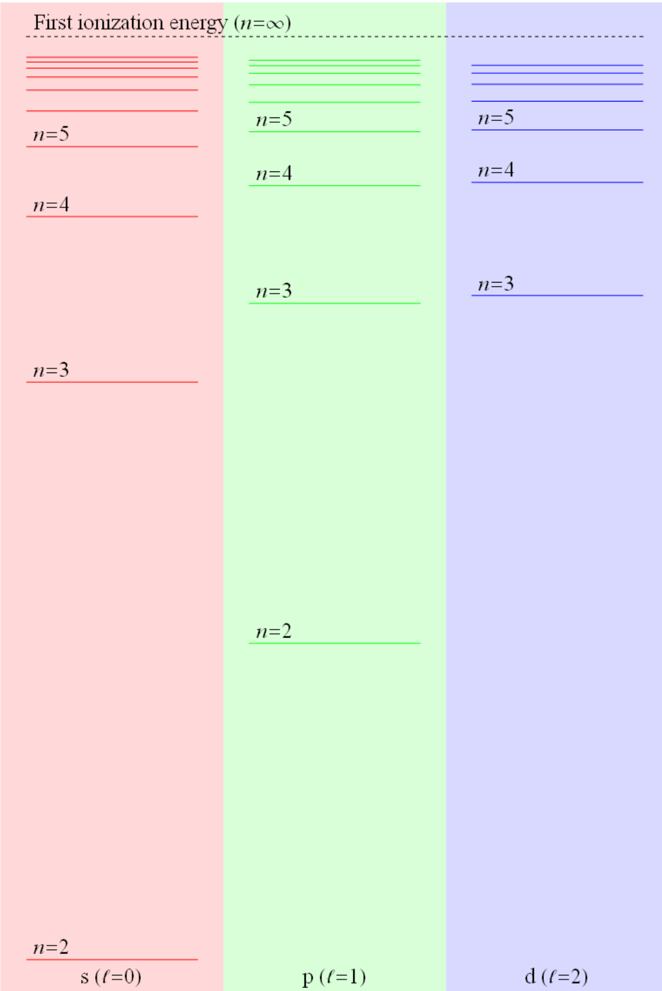
External View



Actual Magnets: Octupole



Highly Excited (Rydberg) Atoms



Energy ~ $1/n^2$

Size ~ n^2

Degeneracy ~ n^2

Degeneracy (cylindrical) ~ n

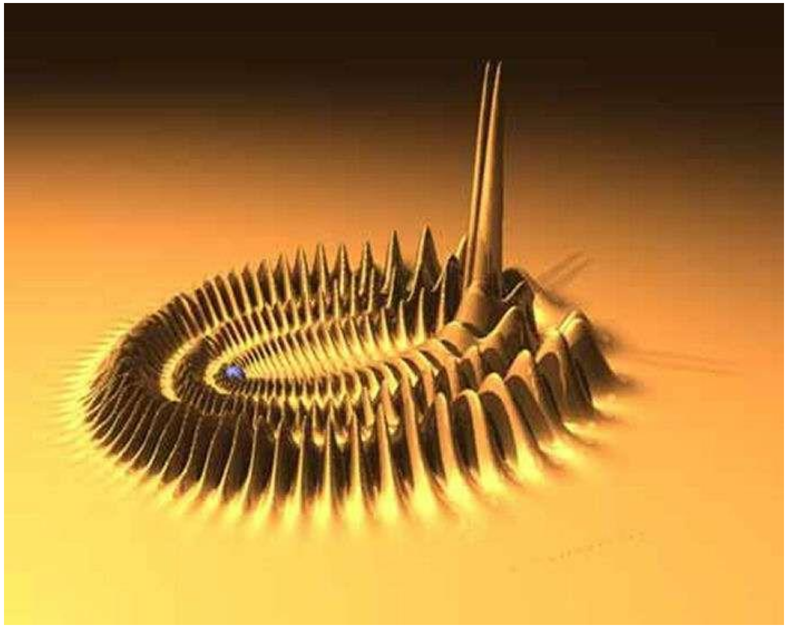
Electric field ~ $1/ n^4$

Magnetic field ~ 1/n

Microwave (n=50 to 51) ~ 51 GHz

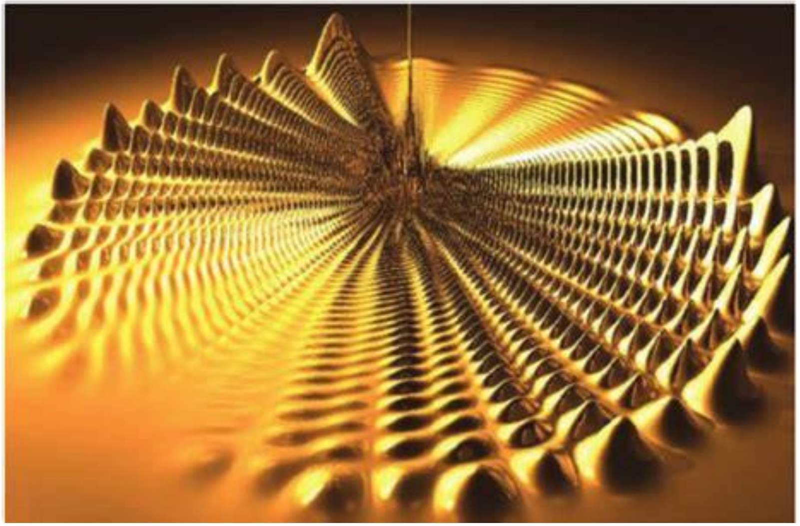
Useful for sensing fields, long range interaction between pairs, strong interaction with a cold ground state atom, …

Trilobite Molecules



Because so many states at nearly same energy, rearrange to get a lot of wave function at a spot

Can bind a cold atom to the Rydberg atom



https://journals.aps.org/prl/abstract/10.1103/PhysRevLett.85.2458

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