# Chapter 4 – The Wavelike Properties of Particles

Louis de Broglie - If light has particle properties, then particles should have wave properties. There must be a wavelength and a frequency.

$$\lambda = \frac{h}{p} \quad \text{and} \quad f = E/h$$

← Energy (what kind?)

$h = 6.626 \times 10^{-34}$ J s

wavelength, momentum, frequency

This turns out to be exactly correct. What to use for E is not a simple algorithm but it turns out that only energy differences are important. (Chap 4 slides 1-8)

Problem on (Chap 4 slide 9)

Na work function $2.28\,eV = 3.65 \times 10^{-19}$ J

$hc/\lambda = 1.99 \times 10^{-25}\ \text{J m} / 450 \times 10^{-9}\ \text{m} = 4.42 \times 10^{-19}$ J

$KE = 4.42 \times 10^{-19}\ \text{J} - 3.65 \times 10^{-19}\ \text{J} = 7.7 \times 10^{-20}$ J

Momentum of electron $p = \sqrt{2\, m\, KE} = \sqrt{2 \cdot 9.11 \times 10^{-31}\ \text{kg}\ 7.7 \times 10^{-20}\ \text{J}} = 3.7 \times 10^{-25}$ kg m/s

(Not relativistic so $KE = p^2/2m$ is OK)

$\lambda = h/p = 6.626 \times 10^{-34}\ \text{J s} / 3.7 \times 10^{-25}\ \text{kg m/s} = 1.8 \times 10^{-9}\ \text{m} = 1.8$ nm

Is there a limit to the size of things that have a wave length? (Chap 4 slide 10)

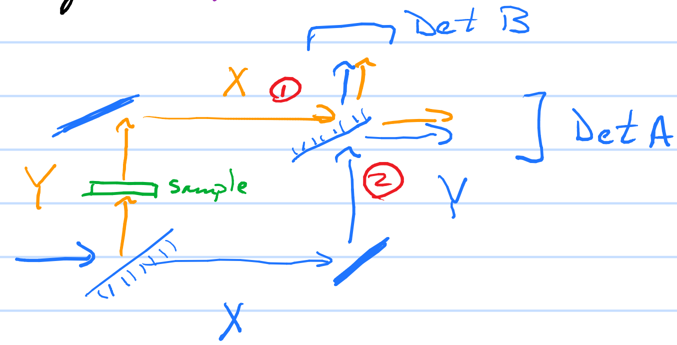
From the $C_{60}$ Nature paper $v = 220$ m/s $\Rightarrow$ $\lambda = 2.5\ \text{pm} = 2.5 \times 10^{-12}$ m

$\lambda = 6.626 \times 10^{-34}\ \text{J s} / (60 \times 12 \times 1.66 \times 10^{-27}\ \text{kg}\ 220\ \text{m/s}) = 2.52 \times 10^{-12}$ m ✓

Everything we can measure $\lambda$ has $\lambda = h/p$ !

One use is to make an atom interferometer and use as a gyroscope. To see how it works, go back to late 19th/early 20th century (Chap 4 Slide 11)

Mach Zehder Interferometer light




① $kY + 2\pi \frac{\Delta\lambda}{\lambda} + kX$

② $kX + kY$

Det A (no sample) has same path

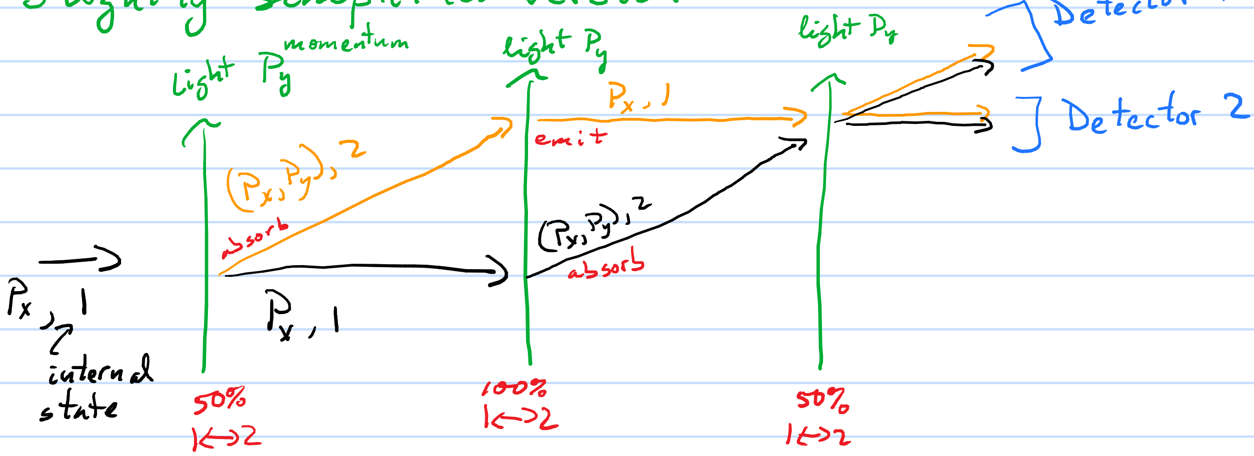
Det B (no sample) has extra $\lambda/2$

Intensity in Det A $= ?\ \cos^2\left(\frac{\pi\,\Delta\lambda}{\lambda}\right)$

" " " B $= ?\ \sin^2\left(\frac{\pi\,\Delta\lambda}{\lambda}\right)$

Is there a way of doing this for atoms? Yes (Chap 4 Slide 12)

Slightly simplified version



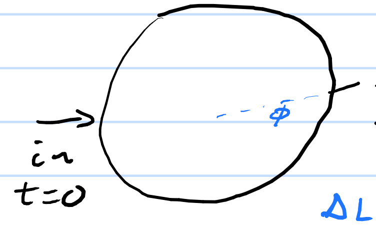

Atoms are harder to work with than light. Why use atoms? Shorter wavelengths!

Visible light $\lambda \sim$ few 100 nm

Atom (Gustavson et al Cs at 290 m/s) $\lambda = \frac{6.626 \times 10^{-34}\,\mathrm{J\,s}}{133 \cdot 1.66 \times 10^{-27}\,\mathrm{kg}\ 290\,\mathrm{m/s}} = 1.0 \times 10^{-11}\,\mathrm{m}$

How can you use this as a gyroscope? If rotating then change the relative lengths of the two paths. Simplify by looking at a circle

$\Delta L =$ difference in path length $= 0$




$\Delta L = 2\phi R$

For light $t = R/c$ $\rightarrow$ $\Delta L = \frac{2R^2}{c}\omega = \frac{2A}{\pi c}\omega$
For matter $t = R/v$ $\Delta L = \frac{2A}{\pi v}\omega$

Measure $\Delta L$ using interference (accuracy $\sim \frac{\lambda}{100}$ to $\frac{\lambda}{10}$

Light can be competitive for gyroscope because easy to use and can send light through looped fiber optic cable.

To understand physics of small objects, we need to understand waves.

Most general wave properties
There is a property of the wave that depends on position and time: $h(x,t)$, $P(\vec{r},t)$, $\vec{E}(\vec{r},t)$, $\vec{B}(\vec{r},t)$
height of string, air pressure, electric/magnetic field

For simplicity, only have $x, t$

$$C(x,t)\,y(x,t) + D(x,t)\frac{\partial y(x,t)}{\partial t} + E(x,t)\frac{\partial y(x,t)}{\partial x} + F(x,t)\frac{\partial^2 y(x,t)}{\partial t^2} + \ldots = 0$$

If $y_1(x,t)$ and $y_2(x,t)$ are solutions, then $y(x,t) = A_1 y_1(x,t) + A_2 y_2(x,t)$ is a solution.

How high can derivatives go? What does $\frac{\partial y(x,t)}{\partial x}$ mean?
Not all C, D, E... gives waves (sometimes diffusion and other phenomena)

What happens if the medium doesn't depend on position or time?

$C(x,t),\ D(x,t), \ldots \rightarrow C, D, \ldots$

Waves need to look like $y(x,t) = y_0 \cos(kx - \omega t)$ or $y_0 \sin(kx - \omega t)$

Use $e^{i\theta} = \cos\theta + i\sin\theta$ to easily incorporate both possibilities

What is a general solution? $y = y_0 e^{i(kx - \omega t)}$
↑ can be complex

$$C y(x,t) + D \frac{\partial y(x,t)}{\partial t} + E \frac{\partial y(x,t)}{\partial x} + F \frac{\partial^2 y(x,t)}{\partial t^2} + G \frac{\partial^2 y(x,t)}{\partial x^2} + \ldots = 0$$

Gives

$$C + -i\omega D + ikE - \omega^2 F - k^2 G + \ldots = 0$$

The $y_0$ is <u>not</u> determined by the equations.

$k = \frac{2\pi}{\lambda}$ $\qquad \omega = 2\pi f$

The wave equation gives the relation between $k$ and $\omega$ or ($\lambda$ and $f$).

For light in vacuum, the only nonzero terms are $F$ and $G$

$\omega^2 = -\frac{G}{F} k^2 \rightarrow \omega = \sqrt{-\frac{G}{F}}\, k \rightarrow 2\pi f = \sqrt{-\frac{G}{F}} \frac{2\pi}{\lambda}$ $\quad \boxed{\sqrt{-\frac{G}{F}} = c}$

For light $\omega = ck$

In general, think of $\omega$ as a function of $k$

$\omega(k)$ – dispersion relation (Chap 4 Slide 13)

Solve $\frac{\partial^2 y}{\partial t^2} - v^2 \frac{\partial^2 y}{\partial x^2} = 0$ $\qquad y = y_0 e^{i(kx - \omega t)}$ (or sin or cos)

$(-i\omega)^2 y - v^2 (ik)^2 y = 0$ $\qquad \omega = \pm vk \Rightarrow y = y_0 e^{ik(x \pm vt)}$

Actually any $g(x \pm vt)$ works (Chap 4 slide 14)
and movie

What happens when you add waves with different $k$?

$$y = A\left\{\cos\left[\left(\bar{k}+\frac{\Delta k}{2}\right)x - \left(\bar{\omega}+\frac{\Delta\omega}{2}\right)t\right] + \cos\left[\left(\bar{k}-\frac{\Delta k}{2}\right)x - \left(\bar{\omega}-\frac{\Delta\omega}{2}\right)t\right]\right\}$$

So complicated, define $\Phi = \bar{k}x - \bar{\omega}t \qquad \Delta\Phi = \Delta k\, x - \Delta\omega\, t$

$y = A\left\{\cos\left[\Phi + \frac{\Delta\Phi}{2}\right] + \cos\left[\Phi - \frac{\Delta\Phi}{2}\right]\right\}$ use trig

$$= 2A \underbrace{\cos(\Phi)}_{\text{fast variation}} \underbrace{\cos(\Delta\Phi/2)}_{\text{slow variation}}$$

If you keep $\Phi = \text{const} \Rightarrow \bar{k}x - \bar{\omega}t = \text{const} \rightarrow x = \text{const} + \frac{\bar{\omega}}{\bar{k}}t$

$v_p$ = phase velocity $= \bar{\omega}/\bar{k}$

If you keep $\Delta\Phi = \text{const} \Rightarrow \Delta k\, x - \Delta\omega\, t = \text{const} \rightarrow x = \text{const} + \frac{\Delta\omega}{\Delta k}t$

$v_g$ = group velocity $= \Delta\omega/\Delta k = \frac{d\omega}{dk}$ (Chap 4 Slides 15-17 + movies)

(Chap 4 slide 18) Width from $\Delta\Phi(x_2) - \Delta\Phi(x_1) = 2\pi$

What happens if you add a bunch of waves? (Chap 4 slides 19-23)

How does this relate to quantum mechanics of nonrelativistic objects? (Chap 4 slide 24)

Which movie is most like light in vacuum?
" " " " nonrelativistic objects?

$\hbar = h/2\pi = 1.0545... \times 10^{-34}$ Js Planck's constant (h-bar)

The relations between $\Delta x, \Delta k$ and $\Delta t, \Delta\omega$ has implications for measurements on particles

Heisenberg uncertainty relations

$\Delta x\, \Delta p_x \geq \hbar/2$

$\Delta t\, \Delta E \geq \hbar/2$

An important aspect is that (for example) $\Delta x \gtrsim \lambda$

Example - Electron microscope (Chap 4, slide 25)

For nonrelativistic speeds $\frac{p^2}{2m} = e \cdot V$ (e: charge, V: voltage)

$$\lambda = h/p = \frac{h}{\sqrt{2meV}}$$

$V = 200\ V \qquad \frac{1}{2}mv^2 = eV \qquad v = \sqrt{\frac{2eV}{m}} = \sqrt{\frac{2 \cdot 1.6\times10^{-19}\, 200}{9.11\times10^{-31}}}\ \text{m/s} = 8.4\times10^{6}\ \frac{m}{s}$

$$\lambda = \frac{h}{mv} = \frac{6.626\times10^{-34}}{9.11\times10^{-31}\ 8.4\times10^{6}}\ m = 8.7\times10^{-11}\ m$$

Higher voltages might need relativity.

Example LIGO mirror $M = 40\ kg \qquad \Delta x = 10^{-4}$ width of proton $\sim 1.7\times10^{-19}\ m$

$\Delta p \sim \frac{1.05\times10^{-34}\ Js}{2 \cdot 1.7\times10^{-19}\ m} \sim 3.1\times10^{-16}\ kg\ m/s \qquad \Delta v = \frac{\Delta p}{m} = 7.7\times10^{-18}\ m/s$

$E \sim \frac{\Delta p^2}{2m} = 1.2\times10^{-33}\ J$ If perfect measurements

$\sim 10^{-10}\ K\ k_B$

center of mass temperature actually $\sim 1\ \mu K = 10^{-6}\ K$

What is meant by $\Delta x$ and $\Delta p$? Standard deviation

$$\sigma_A^2 = \Delta A^2 = \langle A^2\rangle_{average} - (\langle A\rangle_{average})^2 = \frac{1}{N}(A_1^2 + A_2^2 + \dots A_N^2) - \left(\frac{A_1 + A_2 + \dots A_N}{N}\right)^2$$

Suppose measure $x_1 = 1\ m,\ x_2 = 1\ m,\ x_3 = 1\ m,\ x_4 = -2\ m$

$\langle x\rangle_{avg} = \frac{1}{4}(1 + 1 + 1 - 2)\,m = \frac{1}{4}\ m$

$\langle x^2\rangle_{avg} = \frac{1}{4}(1^2 + 1^2 + 1^2 + 2^2)\,m^2 = \frac{7}{4}\ m^2$

$$\Delta x = \sqrt{\frac{7}{4}m^2 - (\frac{1}{4}m)^2} = 1.3\ m$$

The uncertainty relation implies the quantum lowest energy is always higher than the classical lowest energy. Use the harmonic oscillator as example

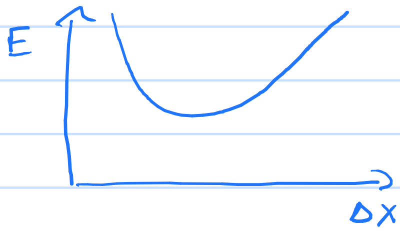
$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2$$

Classical smallest energy $x=0,\ p=0\ ,\ E=0$

Quantum energy $\langle P\rangle_{avg}=0$ $\langle X\rangle_{avg}=0$ $\langle p^2\rangle_{avg} = \Delta p^2$

$\langle X^2\rangle_{avg} = \Delta X^2$

$$E = \frac{\Delta p^2}{2m} + \frac{1}{2}m\omega^2\Delta X^2 \sim \frac{\hbar^2}{8m\Delta X^2} + \frac{1}{2}m\omega^2\Delta X^2$$




How to find the smallest value?

$$\frac{dE}{d\Delta X} = 0 = -\frac{\hbar^2}{4m\Delta X^3} + m\omega^2\Delta X \Rightarrow \Delta X^4 = \frac{\hbar^2}{4m^2\omega^2}$$

$$\boxed{\Delta X^2 = \frac{\hbar}{2m\omega}}$$

$$E = \frac{\hbar^2\, 2m\omega}{8m\,\hbar} + \frac{1}{2}m\omega^2\frac{\hbar}{2m\omega} = \frac{1}{4}\hbar\omega + \frac{1}{4}\hbar\omega = \boxed{\frac{1}{2}\hbar\omega}$$

One way is to use lasers to levitate silica nanospheres and cool the motion. (Chap 4 slide 26)

From Delic et al $\omega = 2\pi\ 315\ kHz = 1.98\times10^{6}\ rad/s$

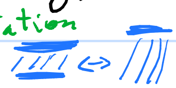
$E = \frac{1}{2}\ 1.05\times10^{-34}\ Js\ \ 1.98\times10^{6}\ rad/s = 1.04\times10^{-28}\ J = 7.5\,\mu K\ k_B$

$M = \frac{4}{3}\pi\left(\frac{143\times10^{-9}m}{2}\right)^3 \cdot 2.2\times10^{3}\frac{kg}{m^3} = 3.4\times10^{-18}\ kg \sim 2\times10^{9}\ amu$

about $10^8$ atoms!

June 2021 LIGO 40 kg mirrors are within factor of ~8 of the lowest possible energy

Many groups have cooled paddle type oscillators to the lowest energy. Whole slab moves, also dilatation



Cleland group (Univ of Chicago)

use higher frequencies so not as cold

$\omega = 2\pi\ 6.1\times10^{9}\ Hz$

$E_{min} \sim \frac{1}{2}\hbar\omega \sim 2\times10^{-24}\ J \sim 0.15\ K\ k_B$

about $20\,\mu m \times 20\,\mu m \times \frac{3}{4}\,\mu m$

$M \sim \frac{1}{2}\ 2000\frac{kg}{m^3}(20\times10^{-6}m)^2\frac{3}{4}\times10^{-6}m$

$\sim 3\times10^{-13}\ kg$