Constrained fits.

Example: Minimize $f(x,y) = x$ subject to the constraint

$$g(x,y) = (x-x_0)^2 + (y-y_0)^2 - R^2 = 0$$

Method of Lagrange Multipliers:

Find stationary points of the function

$$\mathcal{L} = f(x,y) - \lambda g(x,y) = 0$$

with respect to x, y, λ.

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = \frac{\partial \mathcal{L}}{\partial \lambda} = 0 .$$

$$\frac{\partial \mathcal{L}}{\partial x} = 1 - 2\lambda(x-x_0) = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} = -2\lambda(y-y_0) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = R^2 - (x-x_0)^2 - (y-y_0)^2$$

Solution: $y = y_0$

$$x - x_0 = \frac{1}{2\lambda}$$

$$R^2 - \frac{1}{4\lambda^2} = 0 \qquad \lambda = \pm\frac{1}{2R}$$

$$x = x_0 \pm R .$$

as expected.

In general suppose we have measurements $\hat{y}$ with covariance matrix V

$$\chi^2 = (\hat{y} - \vec{y})^T V^{-1} (\hat{y} - \vec{y})$$

is minimized when $\hat{y} = \vec{y}$. What if we impose constraints

$$\vec{g}(\vec{y}, \vec{p}) = 0$$

$\vec{p}$ are parameters that can be expressed as functions of $\vec{y}$.

Example: $\vec{y} = (p_x, p_y, p_z)$

$$\vec{g} = \begin{pmatrix} p_x^2 + p_y^2 + p_z^2 - P^2 \\ E^2 - P^2 - M^2 \end{pmatrix} = 0$$

This introduces two constraints and two parameters (P and E).

$$L = (\hat{y} - \vec{y})^T V^{-1} (\hat{y} - \vec{y}) + 2\vec{\lambda}^T \vec{g}(\vec{y}, \vec{p})$$

$\vec{\lambda}$ is a vector of Lagrange multipliers.

$\vec{g}$ can be linearized about a point $\vec{y}_0, \vec{p}_0$:

$$\vec{g}(\vec{y}, \vec{p}) = \vec{g}(\vec{y}_0, \vec{p}_0) + A(\vec{y} - \vec{y}_0) + B(\vec{p} - \vec{p}_0)$$

$$A = \left.\frac{\partial \vec{g}}{\partial \vec{y}}\right|_{\vec{y}_0, \vec{p}_0} \qquad B = \left.\frac{\partial \vec{g}}{\partial \vec{p}}\right|_{\vec{y}_0, \vec{p}_0}$$

Then,
$$\begin{aligned} L &= \vec{e}^T V^{-1} \vec{e} + 2\vec{\lambda}^T (\vec{g}_0 + A(\vec{y} - \vec{y}_0) + B(\vec{p} - \vec{p}_0)) \\ &= \vec{e}^T V^{-1} \vec{e} + 2\vec{\lambda}^T (\vec{g}_0 - A(\hat{y} - \vec{y}) + A(\hat{y} - \vec{y}_0) + B(\vec{p} - \vec{p}_0)) \\ &= \vec{e}^T V^{-1} \vec{e} + 2\vec{\lambda}^T (\vec{g}_0 + A(\hat{y} - \vec{y}_0) - A\vec{e} + B\vec{d}) \end{aligned}$$

$$\vec{e} = (\hat{y} - \vec{y}) \qquad \vec{d} = \vec{p} - \vec{p}_0$$

$$\frac{\partial L}{\partial \vec{e}} = 2\vec{e}^T V^{-1} - 2\vec{\lambda}^T A = 0$$

$$\frac{\partial L}{\partial \vec{d}} = 2\vec{\lambda}^T B = 0$$

Thus, $\vec{e} = VA^T\vec{\lambda}$

$B^T\vec{\lambda} = 0$

Solve for $\vec{\lambda}$:

$$g(\vec{y},\vec{p}) = \vec{g}_0 + A(\hat{y} - \vec{y}_0) - A\vec{e} + B\vec{d} = 0$$

$$= \vec{g}_0 + A(\hat{y} - \vec{y}_0) - AVA^T\vec{\lambda} + B\vec{d} = 0$$

$$\vec{\lambda} = (AVA^T)^{-1}\left(\vec{g}_0 + A(\hat{y} - \vec{y}_0) + B\vec{d}\right)$$

$$\equiv W\left(\vec{g}_0 + A(\hat{y} - \vec{y}_0) + B\vec{d}\right)$$

But $B^T\vec{\lambda} = 0$ so solve for $\vec{d}$:

$$B^T W\left(\vec{g}_0 + A(\hat{y} - \vec{y}_0) + B\vec{d}\right) = 0$$

$$\vec{d} = -(B^T W B)^{-1} B^T W(\vec{g}_0 + A(\hat{y} - \vec{y}_0))$$

Then,

$$\vec{e} = VA^T W\left(1 - B(B^T W B)^{-1} B^T W\right)\left(\vec{g}_0 + A(\hat{y} - \vec{y}_0)\right)$$

$$\vec{y}_1 = \hat{y} - \vec{e}$$

$$\vec{p}_1 = \vec{p}_0 + \vec{d}$$

$$Cov(\vec{p},\vec{p}) = (B^T W B)^{-1}$$

$$Cov(\vec{y},\vec{p}) = -VA^T W B(B^T W B)^{-1}$$

$$Cov(\vec{y},\vec{y}) = V - VA^T W A V + VA^T W B(B^T W B)^{-1} B^T W A V$$

Examples of constrained Fits.

First, track parameter representation.

Tracks move in helices in a magnetic field. One (of many) parameterizations:

$$p_T = \frac{A B_z}{|\kappa|}$$

$$A = 1.5 \times 10^{-4} \ \mathrm{GeV/c \cdot cm / kG}$$

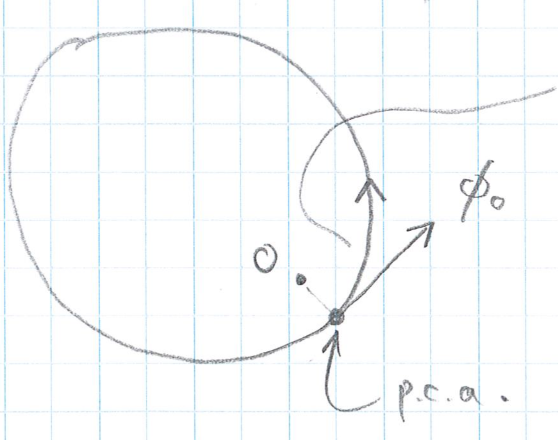
κ is the curvature.

$\kappa = 0$ corresponds to $p_T \to \infty$.

Positively charged particles have $\kappa > 0$
Negatively charged particles have $\kappa < 0$.

$$p_x = p_T \cos \phi_0$$
$$p_y = p_T \sin \phi_0$$
$$p_z = p_T \cot \theta$$

ϕ_0 is the polar angle measured at the point of closest approach to a reference axis, in this case, the z-axis.



ϕ_0 is defined in the direction the track is moving at the p.c.a.

The coordinates of the p.c.a. are

$$x_0 = -d_0 \sin \phi_0$$
$$y_0 = d_0 \cos \phi_0$$

Where $d_0 = -x_0 \sin \phi_0 + y_0 \cos \phi_0$

$= -x_v \sin \phi_0 + y_v \cos \phi_0$ (straight line approx.)

Center of curvature

$$x_c = -(d_0 + 1/2\kappa) \sin \phi_0$$
$$y_c = (d_0 + 1/2\kappa) \cos \phi_0$$

Coordinates along the trajectory :

s = arc length measured from pca.

$$x = x_c + \frac{1}{2\kappa} \sin(\phi_0 + 2\kappa s)$$

$$y = y_c + \frac{1}{2\kappa} \cos(\phi_0 + 2\kappa s)$$

$$z = z_0 + s \cot \theta$$

z_0 = z coordinate at p.c.a.

Straight-line limit :

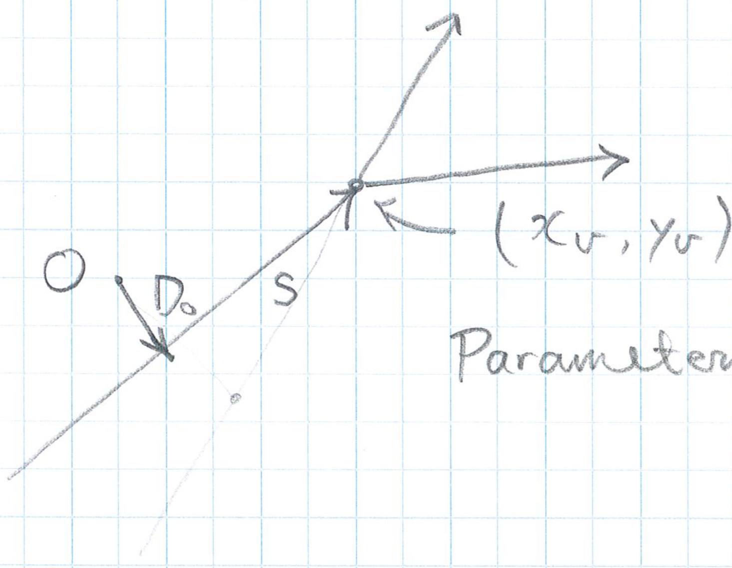
$$x = -d_0 \sin \phi_0 + s \cos \phi_0$$
$$y = d_0 \cos \phi_0 + s \sin \phi_0$$
$$z = z_0 + s \cot \theta \, .$$

Example of a constrained fit.

Two tracks from a common vertex.

- Improves geometric measurements
- Improved ϕ_0 improves p_x, p_y precision

Constraints can introduce convenient parameters.



Parameters
S = decay length
D_0 = impact parameter

$$P_x = p_{x_1} + p_{x_2}$$
$$P_y = p_{y_1} + p_{y_2}$$
$$P_{xy} = \sqrt{P_x^2 + P_y^2}$$

$$x_v = \frac{S P_x - D_0 P_y}{P_{xy}}$$

$$y_v = \frac{S P_y + D_0 P_x}{P_{xy}}$$

Constraints:

$$g_0 = P_x - p_{x_1} - p_{x_2} = P_x - p_{T_1} \cos \phi_{01} - p_{T_2} \cos \phi_{02}$$

$$g_1 = P_y - p_{y_1} - p_{y_2} = P_y - p_{T_2} \sin \phi_{01} - p_{T_2} \sin \phi_{02}$$

$$g_3 = d_{01} + x_v \sin \phi_{01} - y_v \cos \phi_{01}$$

$$g_4 = d_{02} + x_v \sin \phi_{02} - y_v \cos \phi_{02}$$