Assignment #2

1. There are a couple ways to formulate this problem which should all give the same answer.

First, consider the general problem of constrained minimization, but with no additional parameters.

In this case, the function to minimize is

$$L = (\hat{x} - \vec{x})^T V^{-1} (\hat{x} - \vec{x}) + 2\lambda g(\vec{x})$$

where $\hat{x}$ are the measurements with covariance matrix V.

Writing $\vec{x} - \hat{x} = \vec{e}$ and $g(\vec{x}) = g(\hat{x}) + A\vec{e}$

where $A = \left.\frac{\partial g}{\partial \vec{x}}\right|_{\hat{x}}$ gives

$$L = \vec{e}^T V^{-1} \vec{e} + 2\lambda g(\hat{x}) + 2\lambda A \vec{e}$$

Then, $\frac{\partial L}{\partial \vec{e}} = 2\vec{e}^T V^{-1} + 2\lambda A = 0$

$$\vec{e} = -\lambda V A^T$$

But also $A\vec{e} = -g(\hat{x})$

So $-g(\hat{x}) = -\lambda A V A^T$

$$\lambda = g(\hat{x})(AVA^T)^{-1}$$

$$\vec{e} = -VA^T(AVA^T)^{-1} g(\hat{x})$$

In this case,

$$g = z_0 + s\cot\theta - z_0' - s'\cot\theta'$$

$$A = \frac{\partial g}{\partial \vec{x}} = \begin{pmatrix} 1 & s & -1 & -s' \end{pmatrix}$$

$$V = \begin{pmatrix} \sigma_{z_0}^2 & \rho\sigma_{z_0}\sigma_{\cot\theta} & 0 & 0 \\ \rho\sigma_{z_0}\sigma_{\cot\theta} & \sigma_{\cot\theta}^2 & 0 & 0 \\ 0 & 0 & \sigma_{z_0}'^2 & \rho'\sigma_{z_0'}\sigma_{\cot\theta'} \\ 0 & 0 & \rho'\sigma_{z_0'}\sigma_{\cot\theta'} & \sigma_{\cot\theta'}^2 \end{pmatrix}$$

$$VA^T = \begin{pmatrix} \sigma_{z_0}^2 + \rho s\sigma_{z_0}\sigma_{\cot\theta} \\ \rho\sigma_{z_0}\sigma_{\cot\theta} + s\sigma_{\cot\theta}^2 \\ -\sigma_{z_0'}^2 - \rho' s'\sigma_{z_0'}\sigma_{\cot\theta'} \\ -\rho'\sigma_{z_0'}\sigma_{\cot\theta'} - s'\sigma_{\cot\theta'}^2 \end{pmatrix}$$

$$AVA^T = \sigma_{z_0}^2 + 2\rho s\sigma_{z_0}\sigma_{\cot\theta} + s^2\sigma_{\cot\theta}^2 + \sigma_{z_0'}^2 + 2\rho' s'\sigma_{z_0'}\sigma_{\cot\theta'} + s'^2\sigma_{\cot\theta'}^2$$

$$\Delta z_0 = \frac{-\sigma_{z_0}^2 - \rho s\sigma_{z_0}\sigma_{\cot\theta}}{AVA^T}\left(z_0 + s\cot\theta - z_0' - s'\cot\theta'\right)$$

$$\Delta\cot\theta = \frac{-\rho\sigma_{z_0}\sigma_{\cot\theta} - s\sigma_{\cot\theta}^2}{AVA^T}\left(z_0 + s\cot\theta - z_0' - s'\cot\theta'\right)$$

$$\Delta z_0' = \frac{\sigma_{z_0'}^2 + \rho' s'\sigma_{z_0'}\sigma_{\cot\theta'}}{AVA^T}\left(z_0 + s\cot\theta - z_0' - s'\cot\theta'\right)$$

$$\Delta\cot\theta' = \frac{\rho'\sigma_{z_0'}\sigma_{\cot\theta'} + s'\sigma_{\cot\theta'}^2}{AVA^T}\left(z_0 + s\cot\theta - z_0' - s'\cot\theta'\right)$$

then $\vec{x} = \hat{x} + \Delta\vec{x}$.

2. The total expected yield is $Y = S + B$ and the probability of observing N events is

$$P(N|Y) = \frac{e^{-Y} Y^N}{N!} = \frac{e^{-(S+B)} (S+B)^N}{N!}$$

We need to calculate

$$P(S|N,B) = \frac{e^{-(S+B)} (S+B)^N / N!}{\int_0^\infty \frac{e^{-(S+B)} (S+B)^N}{N!} ds}$$

Let $u = S + B$, $du = ds$

$$P(S|N,B) = \frac{e^{-(S+B)} (S+B)^N}{\int_B^\infty e^{-u} u^N du}$$

The denominator can be written in terms of the normalized incomplete gamma function:

$$P(a,x) = \frac{1}{\Gamma(a)} \int_0^x t^{a-1} e^{-t} dt$$

So that $\int_B^\infty e^{-u} u^N du = \Gamma(N+1)\left(1 - P(N+1, B)\right)$

Thus, $P(S|N,B) = \dfrac{e^{-(S+B)} (S+B)^N}{\Gamma(N+1)(1 - P(N+1, B))}$

and we need to solve for $S_{0.95}$ such that

$$\int_0^{S_{0.95}} P(S|N,B)\, ds = 0.95$$

$$\int_0^{S_{0.95}} P(S \mid N, B)\, dS = \frac{\int_B^{B+S_{0.95}} e^{-u} u^N \, du}{\Gamma(N+1)\left(1 - P(N+1), B\right)}$$

$$= \frac{P(N+1, B+S_{0.95}) - P(N+1, B)}{1 - P(N+1, B)}$$

$$= 0.95$$

```
double prob(double *x,double *p) {
  double s = x[0];
  double n = p[0];
  double b = p[1];
  return (TMath::Gamma(n+1,b+s)-TMath::Gamma(n+1,b))/(1-TMath::Gamma(n+1,b));
}

void SignalLimit() {
  double n[] = { 0, 1, 2, 5, 10, 20, 50 };
  TCanvas *c1 = new TCanvas("c1",NULL,0,0,700,900);
  c1->Divide(1,2);
  c1->cd(1);
  gPad->SetGridx();
  gPad->SetGridy();
  TH1F *h = gPad->DrawFrame(0,0,100,1.2);
  h->GetXaxis()->SetTitle("Observed number of events, N");
  h->GetYaxis()->SetTitle("P(S|N,B) with B=0");
  for ( int i=0; i<7; i++ ) {
    TF1 *func = new TF1("func",prob,0,100,2);
    func->SetLineColor(2);
    func->SetParameters(n[i],0);
    func->Draw("same");
  }
  TLine *l = new TLine(0,0.95,100,0.95);
  l->SetLineStyle(2);
  l->Draw();

  c1->cd(2);
  gPad->SetGridx();
  gPad->SetGridy();
  TH1F *h = gPad->DrawFrame(0,0,100,1.2);
  h->GetXaxis()->SetTitle("Observed number of events, N");
  h->GetYaxis()->SetTitle("P(S|N,B) with B=1");
  for ( int i=0; i<7; i++ ) {
    TF1 *func = new TF1("func",prob,0,100,2);
    func->SetLineColor(3);
    func->SetParameters(n[i],1);
    func->Draw("same");
  }
  l->Draw();
}
```

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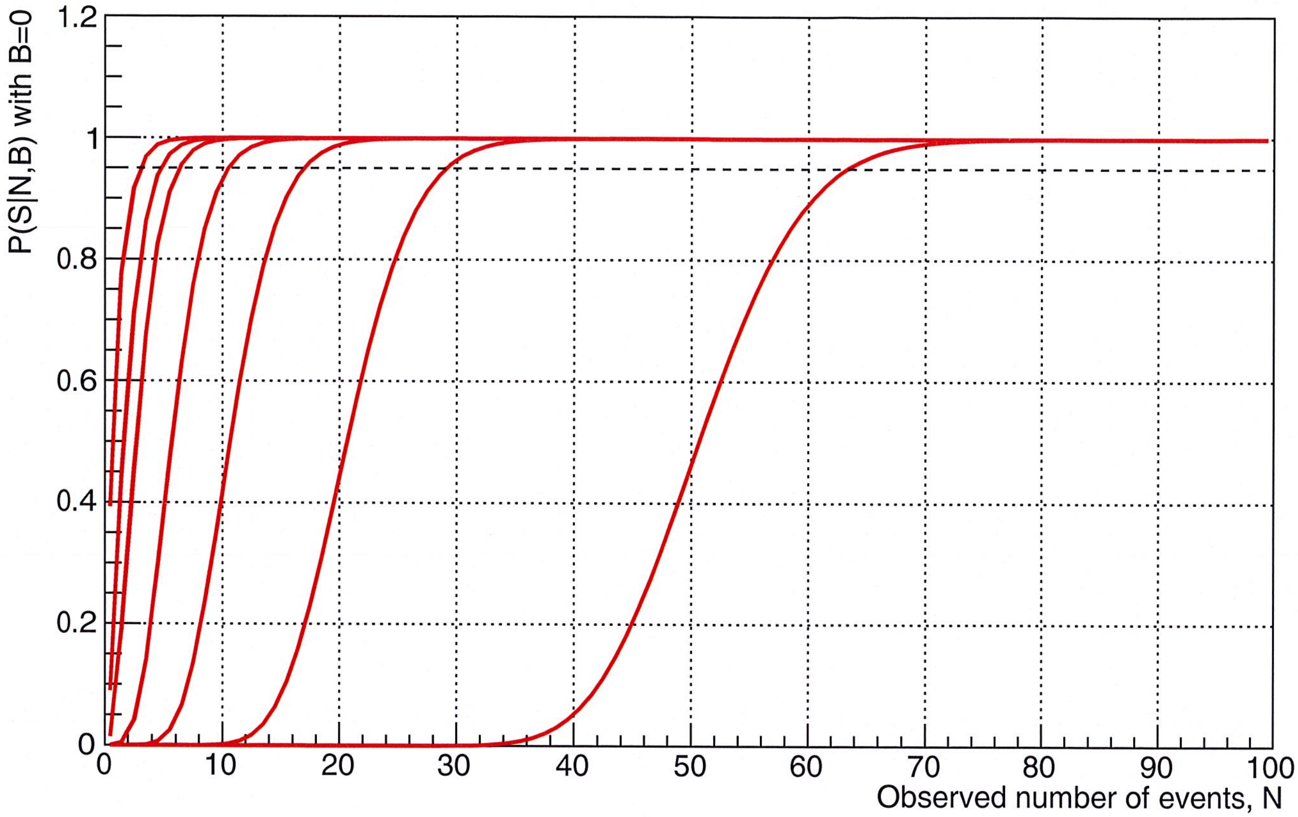
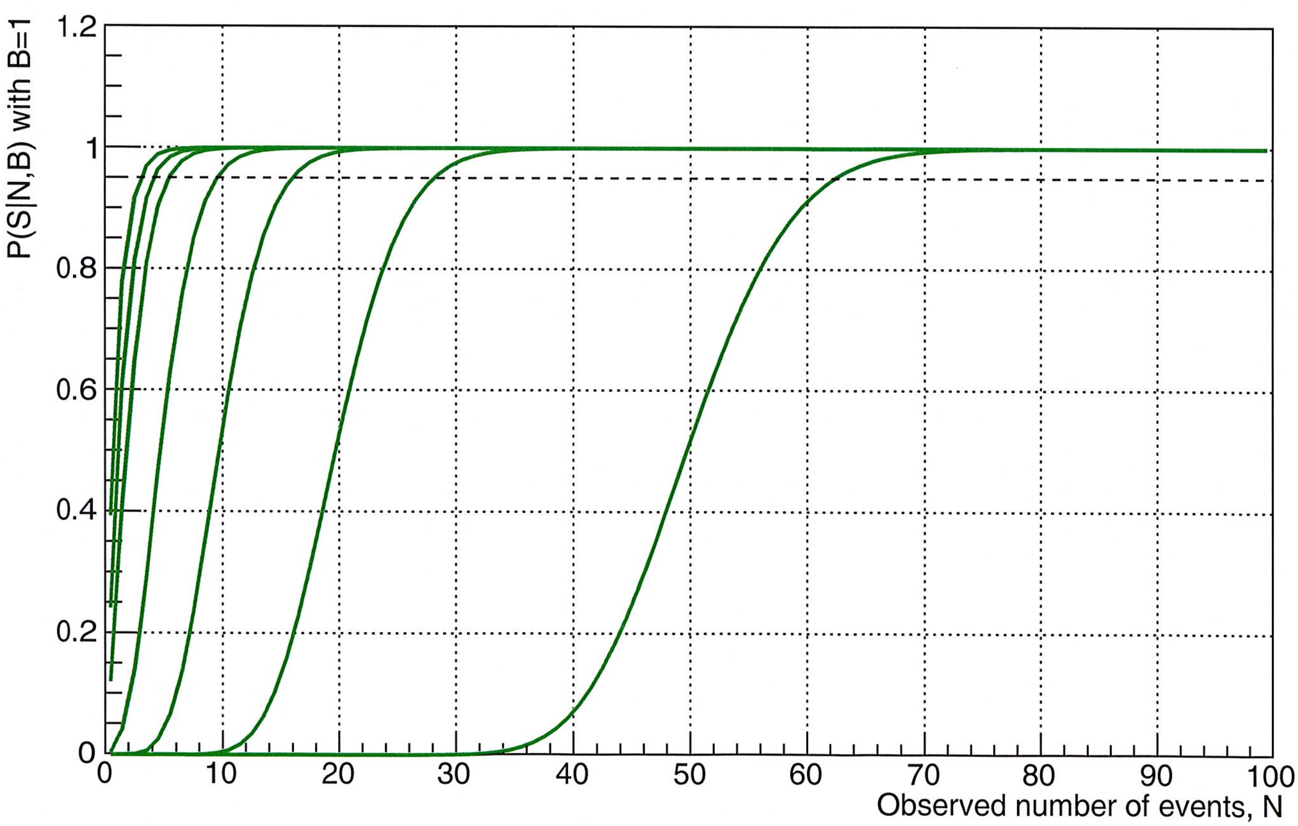


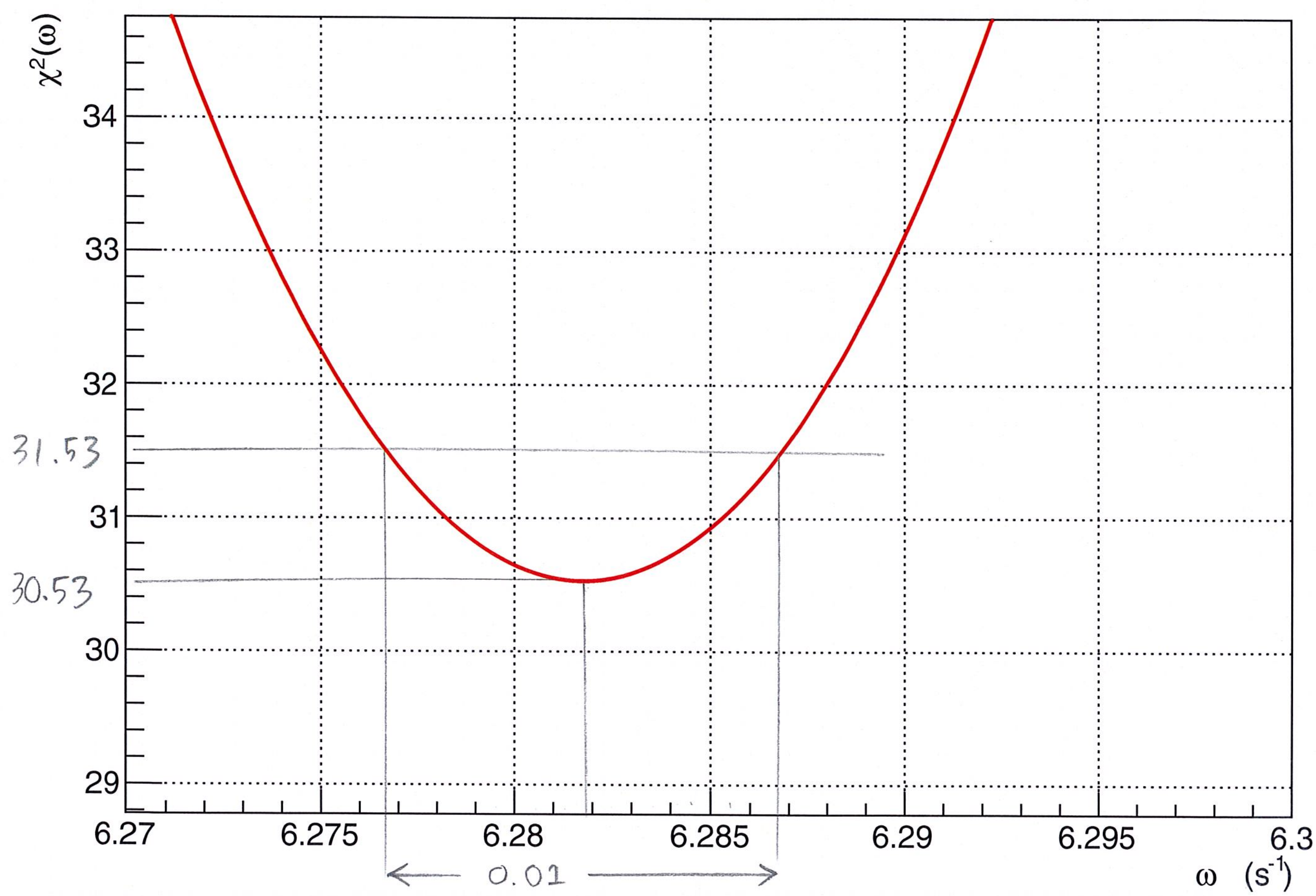
P(S|N,B) with B=0
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
100
Observed number of events, N



P(S|N,B) with B=1
1.2
1
0.8
0.6
0.4
0.2
0
0
10
20
30
40
50
60
70
80
90
100
Observed number of events, N

3(a) Using the Fit method, the value of ω that minimizes the χ^2 statistic is

$$\hat{\omega} = 6.2818 \pm 0.0051 \quad s^{-1}$$



The minimal value of χ^2 is 30.53.

(b) the χ^2 probability density function is

$$f(x;N) = \frac{1}{2^{N/2}\Gamma(N/2)} e^{-x/2} x^{N/2-1}$$

and the probability of observing $x > \chi^2_{min}$ is

$$P(x > \chi^2_{min}) = \int_{\chi^2_{min}}^{\infty} f(x,N)\,dx$$

$$= \frac{2^{-N/2}}{\Gamma(N/2)} \int_{\chi^2_{min}}^{\infty} e^{-x/2} x^{N/2-1}\,dx$$

Let $u = x/2$ $du = \frac{dx}{2}$

Then $P(x > \chi^2_{min}) = \frac{1}{\Gamma(N/2)} \int_{\chi^2_{min}/2}^{\infty} e^{-u} u^{N/2-1}\,du$

$$= 1 - \frac{1}{\Gamma(N/2)} \int_0^{\chi^2_{min}/2} e^{-u} u^{N/2-1}\,du$$

$$= 1 - P(N/2 / \chi^2_{min}/2)$$

where $P(a,x)$ is evaluated using TMath::Gamma.

In this case, $\chi^2_{min} = 30.53$ and $N = 20$ corresponding to

$$P(x > \chi^2_{min}) = 0.0617.$$