Physics 565 - Spring 2011, Assignment #5, Due March 7th

1. Consider the Lagrangian density for a charged scalar field coupled to an electromagnetic field, with additional source terms:

$$\begin{aligned}\mathcal{L} &= (\partial^\mu \phi^*)(\partial_\mu \phi) + m^2\phi^*\phi + \frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) \\ &\quad -ieA^\mu\phi^*\partial_\mu\phi + ieA^\mu(\partial_\mu\phi^*)\phi + e^2A^\mu A_\mu\phi^*\phi \\ &\quad +J^*\phi + J\phi^* + J^\mu A_\mu\end{aligned}$$

(a) If $e \to 0$, the Lagrangian describes independent, free fields coupled to source terms. Using Lagrange's equation, determine the equations of motion for the fields ϕ, ϕ^* and A^μ, in the limit $e \to 0$.
(b) Write the equations of motion in the case when e is not vanishingly small.
(c) Express the fields, $\phi_{(0)}$, $\phi^*_{(0)}$ and $A^\mu_{(0)}$, expanded to zero-th order in e, as integrals over the Green's functions $G(x-x')$, $G^*(x-x')$ and $G^{\mu\nu}(x-x')$.
(d) The following representations can be used to express the Green's functions:

$$\begin{aligned}G_k(x-x') &= \frac{-1}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot(x-x')}}{k^2-m^2+i\epsilon} \\ G^*_k(x-x') &= \frac{-1}{(2\pi)^4}\int d^4k \frac{e^{ik\cdot(x-x')}}{k^2-m^2-i\epsilon} \\ G^{\mu\nu}_k(x-x') &= \frac{-g^{\mu\nu}}{(2\pi)^4}\int d^4k \frac{e^{-ik\cdot(x-x')}}{k^2+i\epsilon}\end{aligned}$$

where the subscript reminds you which variable was used for the momentum integration. Find expressions for $\partial_\mu\phi_{(0)}(x)$ and $\partial_\mu\phi^*_{(0)}(x)$ in terms of the appropriate representations of the Green's functions.
(e) By considering the field $A^\mu_{(1)}(x)$, expanded to first order in e, show that the vertex factor for the following diagram is $-ie(k_1+k_2)_\mu$.