Assignment #7

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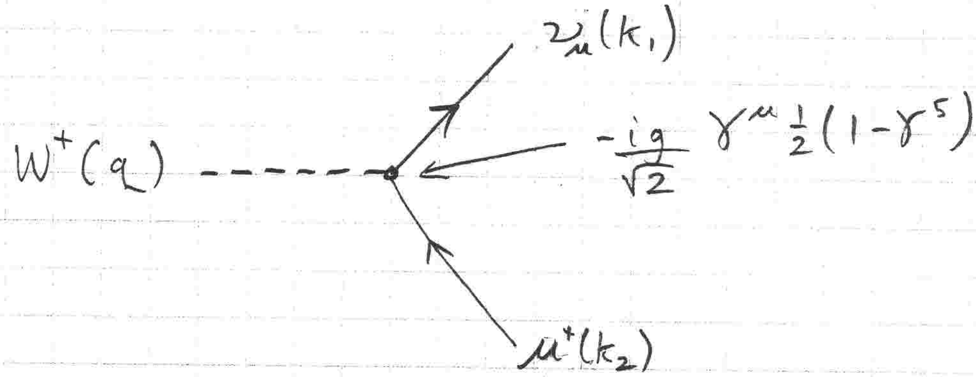
1 (a) Calculate the partial width for $W^+ \to \mu^+ \nu_\mu$.

The Feynman diagram for this process, is



$$-i\mathcal{M} = \bar{u}(k_1)\left(\frac{-ig}{2\sqrt{2}} \gamma^\mu (1-\gamma^5)\right) v(k_2)\, \varepsilon_\mu(q)$$

$$|\mathcal{M}|^2 = \frac{g^2}{8} T^{\mu\nu} \varepsilon_\mu^{(\lambda)}(q)\, \varepsilon_\nu^{(\lambda)*}(q)$$

where
$$\begin{aligned} T^{\mu\nu} &= \sum_{spins} \left(\bar{u}(k_1)\gamma^\mu(1-\gamma^5)v(k_2)\right)\left(\bar{u}(k_1)\gamma^\nu(1-\gamma^5)v(k_2)\right)^\dagger \\ &= \sum_{spins} \mathrm{Tr}\left(u(k_1)\bar{u}(k_1)\gamma^\mu(1-\gamma^5)v(k_2)\bar{v}(k_2)(1+\gamma^5)\gamma^\nu\right) \\ &= \mathrm{Tr}\left(\not{k}_1 \gamma^\mu (1-\gamma^5)\not{k}_2(1+\gamma^5)\gamma^\nu\right) \\ &= 2\,\mathrm{Tr}\left(\not{k}_1\gamma^\mu \not{k}_2 \gamma^\nu (1-\gamma^5)\right) \end{aligned}$$

Also, $\sum_\lambda \varepsilon_\mu^{(\lambda)}(q)\, \varepsilon_\nu^{(\lambda)*}(q) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_W^2}$

Averaging over the three initial polarization states gives

$$\overline{|\mathcal{M}|^2} = \frac{g^2}{3}\left(2k_1\cdot k_2 + \frac{2\,k_1\cdot q\; k_2\cdot q}{M_W^2} - \frac{k_1\cdot k_2\; q\cdot q}{M_W^2}\right)$$

But $q \cdot q = M_W^2$, so

$$|\overline{\mathcal{M}}|^2 = \frac{g^2}{3}\left(k_1 \cdot k_2 + \frac{2 k_1 \cdot q \; k_2 \cdot q}{M_W^2} \right)$$

Now, $k_1 \cdot k_2 = E_1 E_2 + |\vec{k}|^2$
$k_1 \cdot q = M_W E_1$
$k_2 \cdot q = M_W E_2$

so $$|\overline{\mathcal{M}}|^2 = \frac{g^2}{3}\left(E_1 E_2 + |\vec{k}|^2 + 2 E_1 E_2 \right)$$

$$= g^2 \left(E_1 E_2 + \tfrac{1}{3} |\vec{k}|^2 \right)$$

But if we ignore masses of the final state particles, then $E_1 = E_2 = |\vec{k}| = M_W/2$.

Thus, $$|\overline{\mathcal{M}}|^2 = \frac{g^2 M_W^2}{3}.$$

$$d\Gamma = \frac{1}{32\pi^2} |\overline{\mathcal{M}}|^2 \frac{|\vec{k}|}{M_W^2} \, d\Omega$$

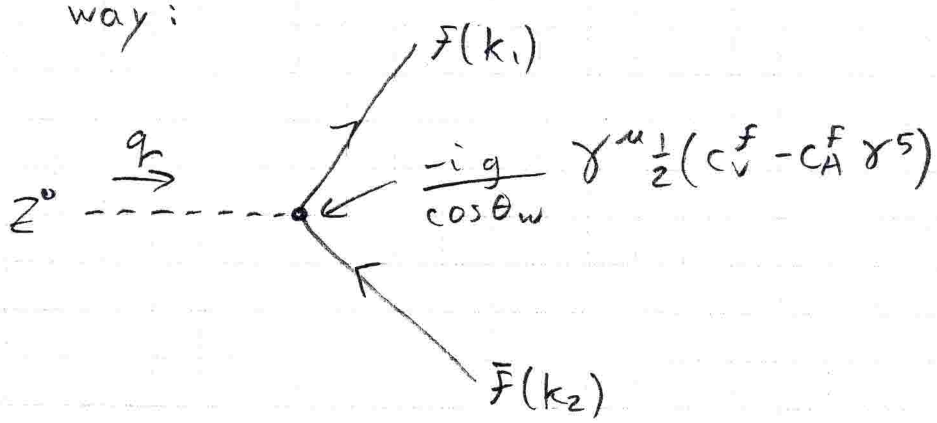
$$= \frac{g^2}{96\pi^2} \cdot \frac{M_W}{2} \, d\Omega .$$

$$\Rightarrow \Gamma = 4\pi \cdot \frac{g^2}{96\pi^2} \cdot \frac{M_W}{2} = \frac{g^2}{48\pi} \, M_W .$$

However, $\frac{G_F}{\sqrt{2}} = \frac{g^2}{8 M_W^2}$ so $\frac{g^2}{8} = \frac{G_F M_W^2}{\sqrt{2}}$

$$\Longrightarrow \Gamma = \frac{G_F M_W^3}{6\sqrt{2}\,\pi} = \frac{G_F \sqrt{2}}{12\pi} \cdot M_W^3 .$$

(b) Ignoring masses of the final state particles, the partial widths for $Z^0 \to \mu^+\mu^-$ and $Z^0 \to \nu_\mu \bar{\nu}_\mu$ can be calculated in a similar way:



Thus,

$$-i\mathcal{M} = \bar{u}(k_1)\left(\frac{-ig}{2\cos\theta_w}\gamma^\mu(c_V^f - c_A^f\gamma^5)\right)v(k_2)\,\varepsilon_\mu(q)$$

$$|\mathcal{M}|^2 = \frac{g^2}{4\cos^2\theta_w}\, T^{\mu\nu}\varepsilon_\mu^{(\lambda)}(q)\varepsilon_\nu^{(\lambda)*}(q)$$

where
$$T^{\mu\nu} = \mathrm{Tr}\left(\not{k}_1\gamma^\mu(c_V^f - c_A^f\gamma^5)\not{k}_2\gamma^\nu(c_V^f - c_A^f\gamma^5)\right)$$
$$= \left[(c_V^f)^2 + (c_A^f)^2\right]\mathrm{Tr}\left(\not{k}_1\gamma^\mu\not{k}_2\gamma^\nu\right) - 2c_V^f c_A^f\,\mathrm{Tr}\left(\not{k}_1\gamma^\mu\not{k}_2\gamma^\nu\gamma^5\right)$$

and
$$\sum_\lambda \varepsilon_\mu^{(\lambda)}(q)\,\varepsilon_\nu^{(\lambda)*}(q) = -g_{\mu\nu} + \frac{q_\mu q_\nu}{M_Z^2}$$

Since this is symmetric in the indices μ and ν, contracting with $T^{\mu\nu}$ only receives contributions from the first term.

$$\Rightarrow |\overline{\mathcal{M}}|^2 = \frac{g^2\left[(c_V^f)^2 + (c_A^f)^2\right]}{12\cos^2\theta_w}\left[4k_1\cdot k_2 + \frac{8\,k_1\cdot q\;k_2\cdot q}{M_Z^2}\right]$$

but again, $k_1 \cdot k_2 = M_Z^2/2$
$k_1 \cdot q = M_Z^2/2$
$k_2 \cdot q = M_Z^2/2$

so $$|\overline{m}|^2 = \frac{g^2[(c_V^f)^2 + (c_A^f)^2]}{3\cos^2\theta_W} M_Z^2$$

$$d\Gamma = \frac{1}{32\pi^2} |\overline{m}|^2 \frac{|\vec{k}|}{M_Z^2} d\Omega$$

$$\Gamma = \frac{1}{16\pi} \frac{|\overline{m}|^2}{M_Z}$$

$$= \frac{g^2[(c_V^f)^2 + (c_A^f)^2]}{48\pi \cos^2\theta_W} \cdot M_Z$$

But $\frac{g^2}{8} = \frac{G_F M_W^2}{\sqrt{2}}$ and $M_Z = \frac{M_W}{\cos\theta_W}$

so $$\Gamma = \frac{G_F M_Z^3}{6\pi\sqrt{2}} [(c_V^f)^2 + (c_A^f)^2]$$

$$= \frac{G_F \sqrt{2} M_Z^3}{12\pi} [(c_V^f)^2 + (c_A^f)^2]$$

Using $c_V^\nu = c_A^\nu = 1/2$,

$$\Gamma(Z^0 \to \nu\bar{\nu}) = \frac{G_F \sqrt{2} M_Z^3}{24\pi}$$

Using $c_V^\mu = -\frac{1}{2} + 2\sin^2\theta_W$ and $c_A^\mu = -\frac{1}{2}$

$$\Gamma(Z^0 \to \mu^+\mu^-) = \frac{G_F\sqrt{2}}{48\pi} M_Z^3 \left(1 + (1 - 4\sin^2\theta_W)^2\right)$$

2 (a) Show that the $Z^0 \to f\bar{f}$ vertex factor

$$\frac{-ig}{\cos\theta_W} \gamma^\mu \frac{1}{2}(c_V^f - c_A^f \gamma^5)$$

can also be written

$$\frac{-ig}{\cos\theta_W} \gamma^\mu \left(c_L^f \frac{1}{2}(1-\gamma^5) + c_R^f \frac{1}{2}(1+\gamma^5)\right).$$

We can equate powers of γ^5 as follows:

$$c_V^f = c_L^f + c_R^f$$

$$c_A^f = c_L^f - c_R^f$$

Hence, $c_L^f = \frac{1}{2}(c_V^f + c_A^f)$

and $c_R^f = \frac{1}{2}(c_V^f - c_A^f)$

Since $c_V^f = T_f^3 - 2Q_f \sin^2\theta_W$
$c_A^f = T_f^3$, we have the following:

f	Q_f	T_f^3	c_A^f	c_V^f	c_L^f	c_R^f
ν_e, ν_μ, ν_τ	0	1/2	1/2	1/2	1/2	0
e^-, μ^-, τ^-	-1	$-1/2$	$-1/2$	$-\frac{1}{2} + 2\sin^2\theta_W$	$\sin^2\theta_W - \frac{1}{2}$	$\sin^2\theta_W$
u, c, t	2/3	1/2	1/2	$\frac{1}{2} - \frac{4}{3}\sin^2\theta_W$	$-\frac{2}{3}\sin^2\theta_W + \frac{1}{2}$	$-\frac{2}{3}\sin^2\theta_W$
d, s, b	$-1/3$	$-1/2$	$-1/2$	$-\frac{1}{2} + \frac{2}{3}\sin^2\theta_W$	$\frac{1}{3}\sin^2\theta_W - \frac{1}{2}$	$\frac{1}{3}\sin^2\theta_W$

With $\sin^2\theta_W = .231$, the numerical values are:

f	Q_f	T_f^3	c_A^f	c_V^f	c_L^f	c_R^f
ν_e, ν_μ, ν_τ	0	1/2	0.5	0.5	.5	0
e^-, μ^-, τ^-	-1	$-1/2$	-0.5	$-.038$	$-.269$	.231
u, c, t	2/3	1/2	0.5	.192	.346	$-.154$
d, s, b	$-1/3$	$-1/2$	-0.5	$-.346$	$-.423$	0.077