Assignment #5

1. Show that $F = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$ can be expressed as $F = 4p^*\sqrt{s}$ in the center-of-mass frame and as $F = 4 m_2 p_1^{lab}$ in the lab frame.

In the center of mass frame, the 4-vectors of the beam particles can be written:

$$p_1 = (E_1, \vec{p}^*) \quad \text{where } E_1 = \sqrt{|\vec{p}^*|^2 + m_1^2}$$
$$p_2 = (E_2, -\vec{p}^*) \quad \text{where } E_2 = \sqrt{|\vec{p}^*|^2 + m_2^2}$$

Thus, $(p_1 \cdot p_2) = E_1 E_2 + |\vec{p}^*|^2$

Then,
$$\begin{aligned}
(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= E_1^2 E_2^2 + 2E_1 E_2 |\vec{p}^*|^2 + |\vec{p}^*|^4 - m_1^2 m_2^2 \\
&= 2|\vec{p}^*|^4 + (m_1^2 + m_2^2)|\vec{p}^*|^2 + 2E_1E_2|\vec{p}^*|^2 \\
&= |\vec{p}^*|^2\left((|\vec{p}^*|^2 + m_1^2) + 2E_1E_2 + (|\vec{p}^*|^2 + m_2^2) \right) \\
&= |\vec{p}^*|^2 (E_1^2 + 2E_1E_2 + E_2^2) \\
&= |\vec{p}^*|^2 (E_1 + E_2)^2 \\
&= |\vec{p}^*|^2 s
\end{aligned}$$

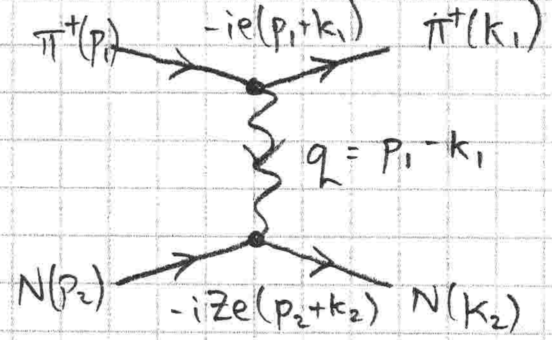
Therefore,
$$\begin{aligned}
F &= 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2} \\
&= 4|\vec{p}^*| \sqrt{s} \quad \text{in the center of mass frame.}
\end{aligned}$$

In the lab frame, we can write $p_1 = (E_1, \vec{p}_1^{lab})$ and $p_2 = (m_2, \vec{0})$ where the target particle with mass m_2 is initially at rest in the lab.

Then, $p_1 \cdot p_2 = E_1 m_2$,
$$\begin{aligned}
(p_1 \cdot p_2)^2 - m_1^2 m_2^2 &= E_1^2 m_2^2 - m_1^2 m_2^2 \\
&= (|\vec{p}_1^{lab}|^2 + m_1^2) m_2^2 - m_1^2 m_2^2 \\
&= m_2^2 |\vec{p}_1^{lab}|^2
\end{aligned}$$

Hence, $F = 4 m_2 |\vec{p}_1^{lab}|$ in the lab frame.

2. Calculate $\frac{d\sigma}{dy}$ for $\pi + Z$ scattering.



Assume the nuclei has charge Z. The invariant amplitude is

$$-i\mathcal{M} = \left(-ie(p_1+k_1)^\mu\right) \frac{g_{\mu\nu}}{q^2} \left(-iZe(p_2+k_2)^\nu\right)$$

$$= -Ze^2 \frac{(p_1+k_1)\cdot(p_2+k_2)}{(p_1-k_1)^2}$$

$$= -Ze^4 \left(\frac{s-u}{t}\right)$$

But $s+t+u = 2m^2 + 2M^2$ so $u = 2m^2+2M^2 - s - t$

Now, $\frac{d\sigma}{dy} = -2EM \frac{d\sigma}{dt}$

and $\frac{d\sigma}{dt} = \frac{1}{64\pi} \frac{1}{s} \frac{|\mathcal{M}|^2}{|\vec{p}^*|^2}$

where $|\mathcal{M}|^2 = Z^2 e^4 \left(\frac{2s - 2m^2 - 2M^2 + t}{t}\right)^2$

and $|\vec{p}^*|^2 = \frac{1}{4s}\left(s - (M+m)^2\right)\left(s - (M-m)^2\right)$ is the momentum of the beam particle in the c.m. frame.

Hence, $$\frac{d\sigma}{dy} = -\frac{2EM}{16\pi} \cdot \frac{1}{(s-(M+m)^2)(s-(M-m)^2)} \cdot Z^2e^4 \left(\frac{2s-2m^2-2M^2+t}{t}\right)^2$$

Next, we must express t in terms of y. Start with

$$t = (p_2 - k_2)^2 \quad \text{in the c.m. frame,}$$

where $p_2 = (M, \vec{0})$ and $k_2 = (E_R, \vec{k})$ where $E_R = M + (E - E')$ is the energy of the recoiling nucleus.

Thus, $$t = p_2^2 + k_2^2 - 2p_2 \cdot k_2$$
$$= 2M^2 - 2ME_R = -2M(E-E')$$

But also, $$s = (p_1 + p_2)^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2$$
$$= m^2 + M^2 + 2ME$$

so $$E = \frac{s - m^2 - M^2}{2M} \quad \text{in the lab frame.}$$

Hence, $$y = \frac{E-E'}{E} = \frac{-t}{2ME} = \frac{-t}{s-m^2-M^2}$$

or $$t = -y(s - m^2 - M^2)$$

So, $$\frac{d\sigma}{dy} = -\frac{EM}{8\pi} \cdot \frac{1}{(s-(M+m)^2)(s-(M-m)^2)}$$
$$\times Z^2e^4 \left(\frac{2s-2m^2-2M^2-y(s-m^2-M^2)}{-y(s-m^2-M^2)}\right)^2$$

where $s = m^2 + M^2 + 2ME$.

(up to this point is sufficient for full credit...)

This can be checked numerically for specific values of E, M, Z.

For example, consider scattering from ^{12}C which has $Z=6$, $M = 12 \times (931.5\ MeV/c^2)$. Suppose $E = 500\ MeV$. Then, using $\alpha = \frac{e^2}{4\pi} = \frac{1}{137}$ the remaining quantities can be calculated:

$$s = m^2 + M^2 + 2ME$$
$$= (140\ MeV)^2 + (12 \times 931.5\ MeV)^2 + 2(12 \times 931.5\ MeV)(500\ MeV)$$
etc.

The fact that $\frac{d\sigma}{dy}$ is negative is a consequence of the change of variables and the limits of integration. The total cross section is positive when integrating from $y=0$ to $y = \frac{E-m}{E}$.

In natural units the cross section has units of $1/(MeV)^2$ which can be expressed in mbarn by multiplying by $(\hbar c)^2 = 0.389 \times 10^6\ MeV^2 \cdot mbarn$.

It is also extremely useful to compare this with the calculation performed in the c.m. frame with the final state momenta boosted into the lab frame so as to calculate y. The results of the two methods should be in exact agreement.

In the center of mass frame,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot |\mathcal{M}|^2 \quad \text{or} \quad \frac{d\sigma}{d(\cos\theta^*)} = \frac{1}{32\pi} \cdot \frac{1}{s} \cdot |\mathcal{M}|^2$$

where $|\mathcal{M}|^2 = Z^2 e^4 \left(\frac{s-u}{t} \right)^2$

and u and t are expressed in terms of $\cos\theta^*$:

$$t = -2|\vec{p}^*|^2 (1 - \cos\theta^*)$$

$$u = m^2 + M^2 - 2E_1E_2 - 2|\vec{p}^*|^2 \cos\theta^*$$

where $E_1 = \sqrt{|\vec{p}^*|^2 + m^2}$ and $E_2 = \sqrt{|\vec{p}^*|^2 + M^2}$

The final 4-momentum of the pion is

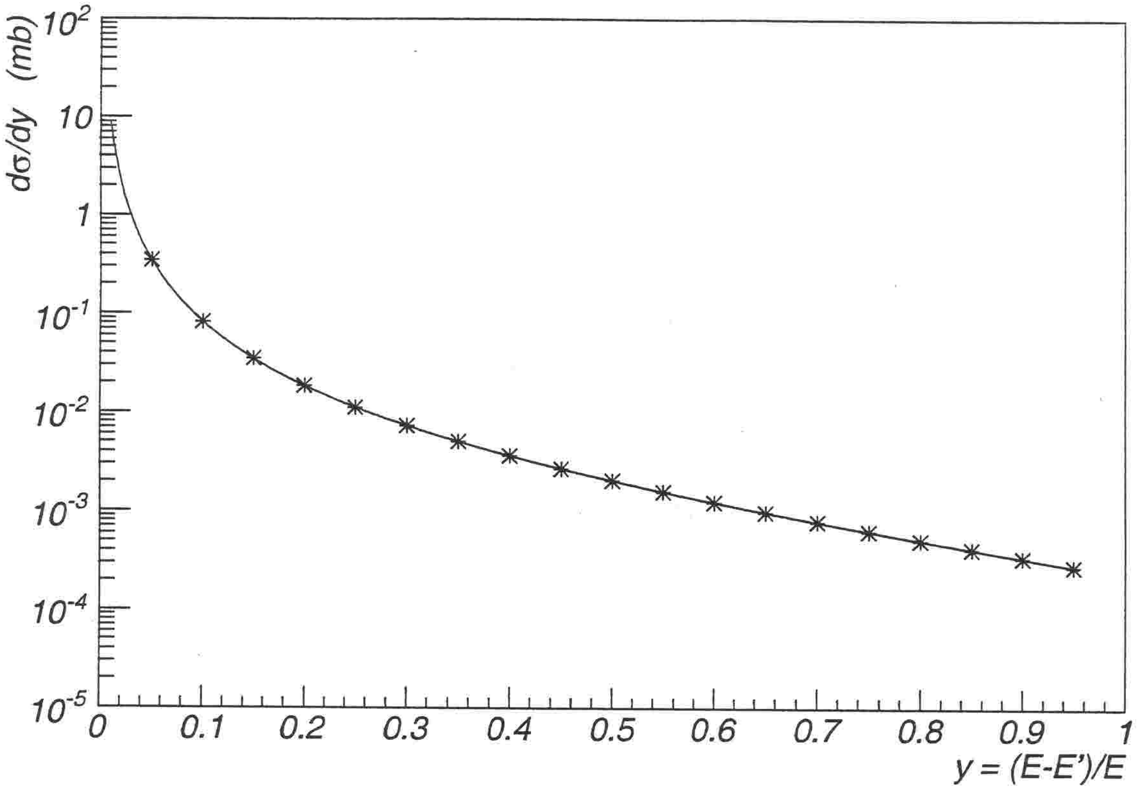
$$k_1 = (E_1', |\vec{p}^*| \sin\theta^* \cos\phi, |\vec{p}^*| \sin\theta^* \sin\phi, |\vec{p}^*| \cos\theta^*)$$

which can be boosted into the lab frame to calculate

$$E' = \gamma E_1' + \gamma\beta |\vec{p}^*| \cos\theta^*$$

which allows one to calculate

$$\frac{d\sigma}{dy} = \frac{d\sigma}{d(\cos\theta^*)} \left(\frac{dy}{d(\cos\theta^*)} \right)^{-1} = \frac{-E}{\gamma\beta|\vec{p}^*|} \frac{d\sigma}{d(\cos\theta^*)}$$

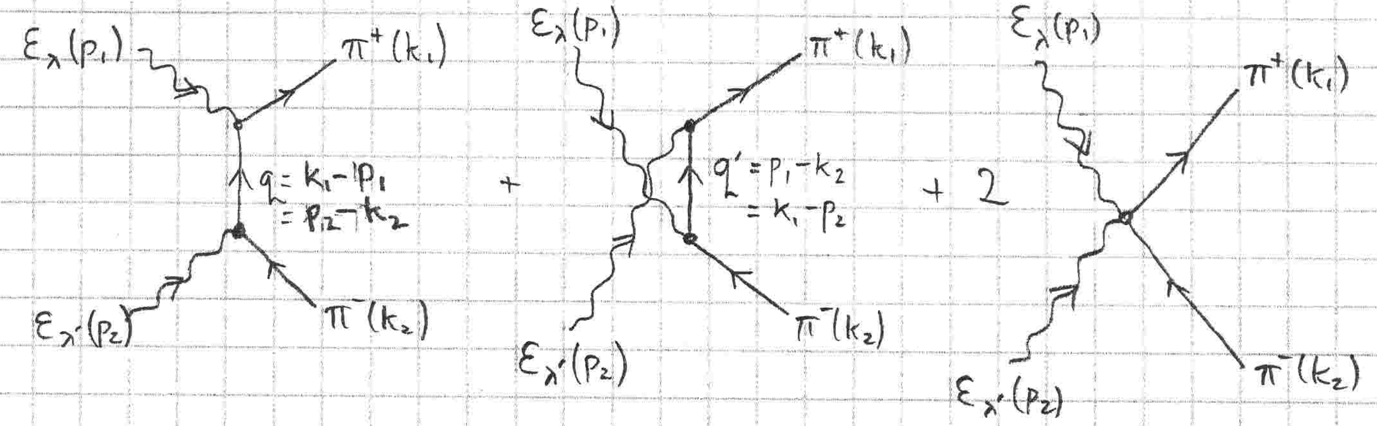


Curve is the calculated differential cross section calculated in the lab frame.

Points are the differential cross section calculated in the cm frame and boosted into the lab frame.

$\Rightarrow$ they agree.

3. Calculate the differential cross section $d\sigma/d\Omega$ for the process with Feynman diagrams:



In the c.m. frame the differential cross section can be written

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{s} \cdot \frac{|\vec{k}|}{|\vec{p}|} \, |\overline{\mathcal{M}}|^2$$

where $|\vec{p}|^2 = \frac{1}{4} s$

and $|\vec{k}|^2 = \frac{1}{4s}\left(s - 4m^2\right) \cdot s = \frac{1}{4} s - m^2$

The remainder of the exercise is to calculate $\mathcal{M}$ and average over the initial photon polarizations.

Recall that the arrows on the pion lines indicate the flow of current in the diagram. Hence, an outgoing π^- with 4-momentum k_2 contributes $-k_2$ at the π-γ vertex. Thus, the first diagram gives

$$-i\mathcal{M}_1 = (\cdot\, e(k_1+q)_\mu)\, \varepsilon^\mu_\lambda(p_1) \frac{1}{q^2-m^2} (\cdot\, e(-k_2+q)_\nu)\, \varepsilon^\nu_{\lambda'}(p_2)$$

where $q = k_1 - p_1 = p_2 - k_2$

$$\text{So, } -i\mathcal{M}_1 = e^2 \frac{(2k_1-p_1)_\mu (p_2-2k_2)_\nu}{(k_1-p_1)^2 - m^2} \varepsilon^\mu_\lambda(p_1)\, \varepsilon^\nu_{\lambda'}(p_2)$$

Likewise, the second diagram gives,

$$-i\mathcal{M}_2 = (\cdot\, e(q'+k_1)_\nu)\, \varepsilon^\nu_{\lambda'}(p_2) \cdot \frac{1}{q'-m^2} (\cdot\, e(-k_2+q')_\mu)\, \varepsilon^\mu_\lambda(p_1)$$

where $q' = p_1 - k_2 = k_1 - p_2$

$$\text{So } -i\mathcal{M}_2 = e^2 \frac{(2k_1-p_2)_\nu (p_1-2k_2)_\mu}{(k_1-p_2)^2 - m^2} \varepsilon^\mu_\lambda(p_1)\, \varepsilon^\nu_{\lambda'}(p_2)$$

while the third diagram is just

$$-i\mathcal{M}_3 = -2e^2 \varepsilon^\mu_\lambda(p_1)\, g_{\mu\nu}\, \varepsilon^\nu_{\lambda'}(p_2)$$

It will be useful to write these as

$$i\mathcal{M} = i\mathcal{M}_1 + i\mathcal{M}_2 + i\mathcal{M}_3 = e^2 T_{\mu\nu}\, \varepsilon^\mu_\lambda(p_1)\, \varepsilon^\nu_{\lambda'}(p_2)$$

$$\text{where } T_{\mu\nu} = \frac{(2k_1-p_1)_\mu (2k_2-p_2)_\nu}{(k_1-p_1)^2 - m^2} + \frac{(2k_1-p_2)_\nu (2k_2-p_1)_\mu}{(k_1-p_2)^2 - m^2} + 2 g_{\mu\nu}$$

Notice, however, that $p_1^\mu T_{\mu\nu}$ and $p_2^\nu T_{\mu\nu}$ both vanish:

$$p_1^\mu T_{\mu\nu} = \frac{(2k_1\cdot p_1)(2k_2-p_2)_\nu}{-2k_1\cdot p_1} + \frac{(2k_1-p_2)_\nu(2k_2\cdot p_1)}{-2k_2\cdot p_1} + 2p_{1\nu}$$

$$= -(2k_2-p_2)_\nu - (2k_1-p_2)_\nu + 2p_{1\nu}$$

$$= 2(p_1+p_2-k_1-k_2)_\nu$$

$$= 0 \quad \text{by momentum conservation.}$$

Similarly,

$$p_2^\nu T_{\mu\nu} = \frac{(2k_1-p_1)_\mu(2k_2\cdot p_2)}{-2k_2\cdot p_2} + \frac{(2k_1\cdot p_2)(2k_2-p_1)_\mu}{-2k_1\cdot p_2} + 2p_{2\mu}$$

$$= -(2k_1-p_1)_\mu - (2k_2-p_1)_\mu + 2p_{2\mu}$$

$$= 2(p_1+p_2-k_1-k_2)_\mu$$

$$= 0, \quad \text{again by momentum conservation.}$$

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$$|\overline{\mathcal{M}}|^2 = \frac{e^4}{4} \sum_{\lambda,\lambda'} \left(T_{\mu\nu}\, \varepsilon^{\mu}_{\lambda}(p_1)\, \varepsilon^{\nu}_{\lambda'}(p_2) \right)^* \left(T_{\rho\sigma}\, \varepsilon^{\rho}_{\lambda}(p_1)\, \varepsilon^{\sigma}_{\lambda'}(p_2) \right)$$

$$= \frac{e^4}{4} \sum_{\lambda,\lambda'} T^*_{\mu\nu} T_{\rho\sigma} \left(\varepsilon^{*\mu}_{\lambda}(p_1)\, \varepsilon^{\rho}_{\lambda}(p_1) \right) \left(\varepsilon^{*\nu}_{\lambda'}(p_2)\, \varepsilon^{\sigma}_{\lambda'}(p_2) \right)$$

But $\sum_{\lambda=\pm 1} \varepsilon^{*\mu}_{\lambda}(p_1)\, \varepsilon^{\rho}_{\lambda}(p_1) = -g^{\mu\rho} + \frac{p_1^{\mu} n^{\rho} + p_1^{\rho} n^{\mu}}{p_1 \cdot n} - \frac{p_1^{\mu} p_1^{\rho}}{(p_1 \cdot n)^2}$

and likewise,

$$\sum_{\lambda'=\pm 1} \varepsilon^{*\nu}_{\lambda'}(p_2)\, \varepsilon^{\sigma}_{\lambda'}(p_2) = -g^{\nu\sigma} + \frac{p_2^{\nu} n^{\sigma} + p_2^{\sigma} n^{\nu}}{p_2 \cdot n} - \frac{p_2^{\nu} p_2^{\sigma}}{(p_2 \cdot n)^2}$$

so the only nonvanishing terms are :

$$|\overline{\mathcal{M}}|^2 = \frac{e^4}{4} T^*_{\mu\nu} T_{\rho\sigma}\, g^{\mu\rho} g^{\nu\sigma} = \frac{e^4}{4} T^*_{\mu\nu} T^{\mu\nu}$$

$$T_{\mu\nu}T^{\mu\nu} = \frac{(2k_1-p_1)^2(2k_2-p_2)^2}{-2k_1\cdot p_1} + \frac{(2k_1-p_2)^2(2k_2-p_1)^2}{-2k_1\cdot p_2} + 8$$

$$+ \frac{(2k_1-p_1)\cdot(2k_2-p_1)\,(2k_2-p_2)\cdot(2k_1-p_2)}{4k_1\cdot p_1\;k_1\cdot p_2}$$

$$+ \frac{(2k_1-p_1)\cdot(2k_2-p_2)}{-2k_1\cdot p_1} + \frac{(2k_1-p_2)\cdot(2k_2-p_1)}{-2k_1\cdot p_2}$$

then, using $p_1^2 = p_2^2 = 0$, $k_1^2 = k_2^2 = m^2$

$k_2\cdot p_2 = k_1\cdot p_1$, $k_1\cdot p_2 = k_2\cdot p_1$

$k_1\cdot k_2 = k_1\cdot(p_1+p_2-k_1) = k_1\cdot p_1 + k_2\cdot p_1 - m^2$

$p_1\cdot p_2 = p_1\cdot(k_1+k_2-p_1) = k_1\cdot p_1 + k_2\cdot p_1$

this can be written

$$T_{\mu\nu}T^{\mu\nu} = 4m^4\left(\frac{1}{(k_1\cdot p_1)^2} + \frac{1}{(k_2\cdot p_1)^2} + \frac{2}{(k_1\cdot p_1)(k_2\cdot p_1)}\right)$$

$$-8m^2\left(\frac{1}{k_1\cdot p_1} + \frac{1}{k_2\cdot p_1}\right) + 8$$

$$= 4m^4\left(\frac{1}{k_1\cdot p_1} + \frac{1}{k_2\cdot p_1}\right)^2$$

$$-8m^2\left(\frac{1}{k_1\cdot p_1} + \frac{1}{k_2\cdot p_1}\right) + 8$$

$$= 4\left[\left(\frac{m^2}{k_1\cdot p_1} + \frac{m^2}{k_2\cdot p_1} - 1\right)^2 + 1\right]$$

So $$|\overline{\mathcal{M}}|^2 = e^4\left[\left(\frac{m^2}{k_1\cdot p_1} + \frac{m^2}{k_2\cdot p_1} - 1\right)^2 + 1\right]$$

Next, we can write the products of 4-vectors in terms of the scattering angle in the center of mass frame:

$$k_1 \cdot p_1 = E^2 - E|\vec{k}|\cos\theta$$
$$k_2 \cdot p_1 = E^2 + E|\vec{k}|\cos\theta$$

and use $E = \frac{\sqrt{s}}{2}$, $|\vec{k}| = \sqrt{E^2 - m^2} = \sqrt{s/4 - m^2}$

So $$k_1 \cdot p_1 = \frac{s}{4} - \frac{\sqrt{s}}{2}\sqrt{\frac{s}{4} - m^2}\,\cos\theta$$
$$= \frac{s}{4}\left(1 - \sqrt{1 - \frac{4m^2}{s}}\cos\theta\right)$$

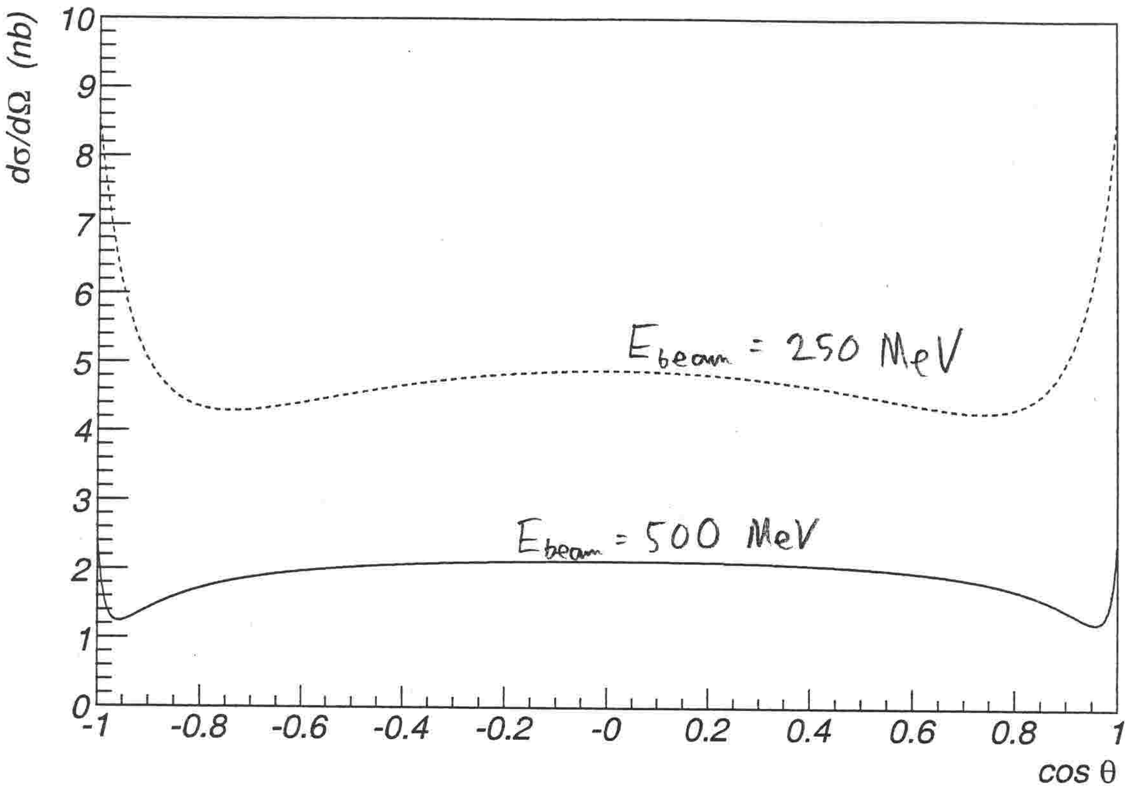
and $$k_2 \cdot p_1 = \frac{s}{4}\left(1 + \sqrt{1 - \frac{4m^2}{s}}\cos\theta\right)$$

Thus, $$|\overline{\mathcal{M}}|^2 = e^4\left[\left(\frac{4m^2/s}{1 - \sqrt{1-4m^2/s}\,\cos\theta} + \frac{4m^2/s}{1 + \sqrt{1-4m^2/s}\,\cos\theta} - 1\right)^2 + 1\right]$$

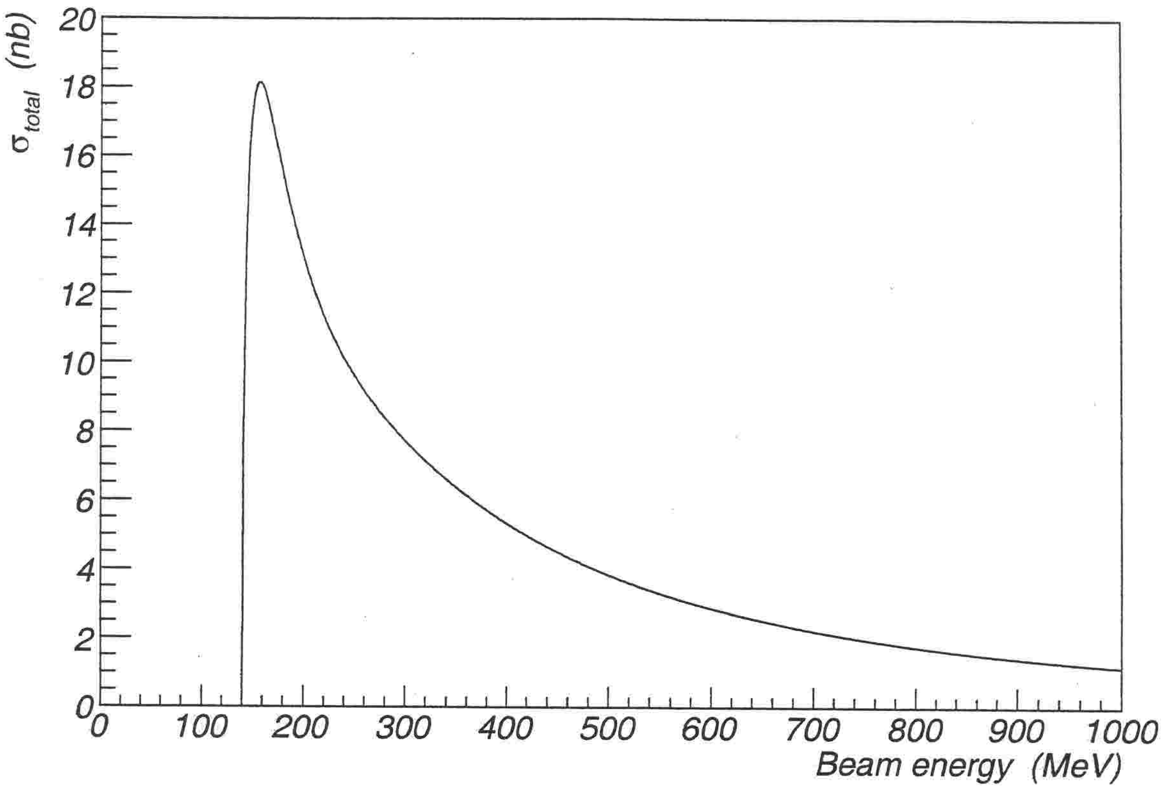
$$\frac{d\sigma}{d\Omega} = \frac{e^4}{64\pi^2}\cdot\frac{1}{s}\sqrt{1 - 4m^2/s}\left(\left(\frac{4m^2/s}{1 - \sqrt{1-4m^2/s}\,\cos\theta} + \frac{4m^2/s}{1 + \sqrt{1-4m^2/s}\,\cos\theta} - 1\right)^2 + 1\right)$$

With $\alpha = \frac{e^2}{4\pi}$,

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16s}\sqrt{1 - 4m^2/s}\left[\left(\frac{4m^2/s}{1 - \sqrt{1-4m^2/s}\,\cos\theta} + \frac{4m^2/s}{1 + \sqrt{1-4m^2/s}\,\cos\theta} - 1\right)^2 + 1\right]$$



dσ/dΩ (nb)
10
9
8
7
6
5
4
3
2
1
0
-1
-0.8
-0.6
-0.4
-0.2
-0
0.2
0.4
0.6
0.8
1
cos θ
E_beam = 250 MeV
E_beam = 500 MeV



Total cross section calculated using numerical integration of $\frac{d\sigma}{d(\cos\theta)}$.