Assignment #4

1. $$\mathcal{L} = (D^\mu \vec{\phi}) \cdot (D_\mu \vec{\phi}) - m^2 \vec{\phi} \cdot \vec{\phi} - \frac{1}{4} \vec{F}^{\mu\nu} \cdot \vec{F}_{\mu\nu}$$

where $$D^\mu \vec{\phi} = \partial_\mu \vec{\phi} + g \vec{W}_\mu \times \vec{\phi}$$
$$\vec{F}_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu + g W_\mu \times W_\nu$$

(a) Under a gauge transformation,

$$\vec{W}_\mu \rightarrow \vec{W}'_\mu = \vec{W}_\mu - \vec{\lambda}(x) \times \vec{W}_\mu + \frac{1}{g} \partial_\mu \vec{\lambda}(x)$$

Consider the terms in the field tensor separately:

$$\partial_\mu \vec{W}_\nu \rightarrow \partial_\mu \vec{W}'_\nu = \partial_\mu \vec{W}_\nu - (\partial_\mu \vec{\lambda}(x)) \times \vec{W}_\nu - \vec{\lambda}(x) \times (\partial_\mu \vec{W}_\nu) + 1/g \; \partial_\mu \partial_\nu \vec{\lambda}(x)$$

$$g \vec{W}_\mu \times \vec{W}_\nu \rightarrow g \vec{W}'_\mu \times \vec{W}'_\nu = g \vec{W}_\mu \times \vec{W}_\nu - g(\vec{\lambda}(x) \times \vec{W}_\mu) \times \vec{W}_\nu - g \vec{W}_\mu \times (\vec{\lambda}(x) \times \vec{W}_\nu) + (\partial_\mu \vec{\lambda}(x)) \times \vec{W}_\nu + \vec{W}_\mu \times (\partial_\nu \vec{\lambda}(x)) + \mathcal{O}(\lambda^2)$$

But recall that $(\vec{A} \times \vec{B}) \times \vec{C} + \vec{B} \times (\vec{A} \times \vec{C}) = \vec{A} \times (\vec{B} \times \vec{C})$ so this can be written:

$$g \vec{W}'_\mu \times \vec{W}'_\nu = g \vec{W}_\mu \times \vec{W}_\nu - g \vec{\lambda}(x) \times (\vec{W}_\mu \times \vec{W}_\nu) + (\partial_\mu \vec{\lambda}(x)) \times \vec{W}_\nu + \vec{W}_\mu \times (\partial_\nu \vec{\lambda}(x)) + \mathcal{O}(\lambda^2)$$

Hence, $\vec{F}_{\mu\nu} \rightarrow \vec{F}'_{\mu\nu} = \partial_\mu \vec{W}'_\nu - \partial_\nu \vec{W}'_\mu + g \vec{W}'_\mu \times \vec{W}'_\nu$

$$= \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu - \cancel{(\partial_\mu \vec{\lambda}) \times \vec{W}_\nu} - \vec{\lambda} \times (\partial_\mu \vec{W}_\nu) + \cancel{(\partial_\nu \vec{\lambda}) \times \vec{W}_\mu} + \vec{\lambda} \times (\partial_\nu \vec{W}_\mu)$$
$$+ g \vec{W}_\mu \times \vec{W}_\nu - g \vec{\lambda} \times (\vec{W}_\mu \times \vec{W}_\nu) + \cancel{(\partial_\mu \vec{\lambda}) \times \vec{W}_\nu} + \cancel{\vec{W}_\mu \times (\partial_\nu \vec{\lambda})} + \mathcal{O}(\lambda^2)$$

Therefore, $\vec{F}'_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g \vec{W}_\mu \times \vec{W}_\nu$

$$- \vec{\lambda} \times \left(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g \vec{W}_\mu \times \vec{W}_\nu \right) + \mathcal{O}($$

So the field tensor $\vec{F}_{\mu\nu}$ transforms in the same way that the fields $\vec{\phi}$ did:

$$\vec{\phi} \rightarrow \vec{\phi}' = \vec{\phi} - \vec{\lambda} \times \vec{\phi}$$

$$\vec{F}_{\mu\nu} \rightarrow \vec{F}'_{\mu\nu} = \vec{F}_{\mu\nu} - \vec{\lambda} \times \vec{F}_{\mu\nu}$$

We know already that $\vec{\phi} \cdot \vec{\phi}$ is invariant and so for the same reason, $\vec{F}_{\mu\nu} \cdot \vec{F}^{\mu\nu}$ must also be invariant.

Therefore, the term $-\frac{1}{4} \vec{F}^{\mu\nu} \cdot \vec{F}_{\mu\nu}$ and hence, the entire Lagrangian must be gauge invariant.

(b) $$\mathcal{L} = (D^\mu \vec{\phi})\cdot(D_\mu \vec{\phi}) - m^2 \vec{\phi}\cdot\vec{\phi} - \frac{1}{4}\vec{F}^{\mu\nu}\cdot\vec{F}_{\mu\nu}$$

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} + g\vec{W}_\mu \times \vec{\phi}$$

$$\vec{F}_{\mu\nu} = \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g\vec{W}_\mu \times \vec{W}_\nu$$

Lowest order terms:

$$\mathcal{L}_0 = (\partial_\mu \vec{\phi})\cdot(\partial^\mu \vec{\phi}) - m^2 \vec{\phi}\cdot\vec{\phi}$$
$$- \frac{1}{4}(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu)\cdot(\partial^\mu \vec{W}^\nu - \partial^\nu \vec{W}^\mu)$$

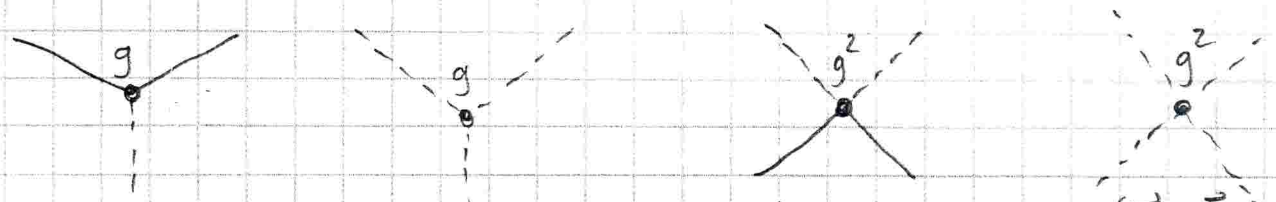
Terms that are first-order in g:

$$\mathcal{L}_g = g(\partial_\mu \vec{\phi})\cdot(\vec{W}^\mu \times \vec{\phi}) + g(\vec{W}_\mu \times \vec{\phi})\cdot(\partial^\mu \vec{\phi})$$
$$- \frac{g}{4}(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu)\cdot(\vec{W}_\mu \times \vec{W}_\nu)$$
$$- g/4\,(\vec{W}^\mu \times \vec{W}^\nu)\cdot(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu)$$
$$= 2g(\partial_\mu \vec{\phi})\cdot(\vec{W}^\mu \times \vec{\phi}) - \frac{g}{2}(\partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu)\cdot(\vec{W}_\mu \times \vec{W}_\nu)$$

Terms of second order in g:

$$\mathcal{L}_{g^2} = g^2(\vec{W}_\mu \times \vec{\phi})\cdot(W^\mu \circ \vec{\phi}) + g^2(\vec{W}_\mu \times \vec{W}_\nu)\cdot(\vec{W}^\mu \times \vec{W}^\nu)$$

Notice that terms of order g have 3 fields while terms of order g^2 have 4 fields. These will correspond to interactions of the form



$(\partial_\mu \vec{\phi})\cdot(\vec{W}^\mu \times \vec{\phi})$ $\quad$ $(\partial_\mu \vec{W}_\nu)\times(\vec{W}_\mu \times \vec{W}_\nu)$ $\quad$ $(\vec{W}_\mu \times \vec{\phi})\cdot(\vec{W}^\mu \times \vec{\phi})$ $\quad$ $(\vec{W}_\mu \times \vec{W}_\nu)\cdot(\vec{W}^\mu \times$

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(c) Consider the term

$$\mathcal{L}_m = m^2 \vec{W}_\mu \cdot \vec{W}^\mu$$

Under a gauge transformation,

$$\vec{W}_\mu \rightarrow \vec{W}_\mu{}' = \vec{W}_\mu - \vec{\lambda}(x) \times \vec{W}_\mu + \frac{1}{g} \partial_\mu \vec{\lambda}(x)$$

So to first order in $\vec{\lambda}$:

$$\mathcal{L}'_m = m^2 \vec{W}_\mu \cdot \vec{W}^\mu - m^2 (\vec{\lambda} \times \vec{W}_\mu) \cdot \vec{W}^\mu - m^2 \vec{W}_\mu \cdot (\vec{\lambda} \times \vec{W}^\mu) + \frac{m^2}{g} \vec{W}_\mu \cdot (\partial^\mu \vec{\lambda}) + \frac{m^2}{g} (\partial_\mu \vec{\lambda}) \cdot \vec{W}^\mu + \mathcal{O}(\lambda^2)$$

Two of the terms vanish because $\vec{A} \cdot (\vec{A} \times \vec{B}) = 0$ but we are still left with

$$\mathcal{L}'_m = m^2 \vec{W}_\mu \cdot \vec{W}^\mu + \frac{2m^2}{g} \vec{W}_\mu \cdot (\partial^\mu \vec{\lambda})$$

there are no other terms of order λ so this expression cannot be gauge invariant unless $m^2 = 0$.