Assignment # 3

1. Consider the current operator

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

(a) Show that $\partial_\mu j^\mu = 0$.

$$\partial_\mu j^\mu = (\partial_\mu \bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu\partial_\mu\psi$$

But $(i\gamma^\mu\partial_\mu - m)\psi = 0$
and $\bar{\psi}(i\overleftarrow{\partial_\mu}\gamma^\mu + m) = 0$

So, $(\partial_\mu\bar{\psi})\gamma^\mu = im\bar{\psi}$

$\gamma^\mu(\partial_\mu\psi) = -im\psi$

So $\partial_\mu j^\mu = im\bar{\psi}\psi - im\bar{\psi}\psi = 0$

(b) Express $Q = \int d^3x\, j^0(x)$ in terms of creation and annihilation operators in normal order.

$$j^0 = \bar{\psi}\gamma^0\psi = \psi^+\psi$$

Using $\psi(x) = \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_k}\sum_{\lambda=1,2}\left[u^{(\lambda)}(k)\,b^{(\lambda)}(k)\,e^{-ik\cdot x} + v^{(\lambda)}(k)\,d^{(\lambda)+}(k)\,e^{ik\cdot x}\right]$

$$\psi^+(x) = \int\frac{d^3k}{(2\pi)^3}\frac{m}{\omega_k}\sum_{\lambda=1,2}\left[u^{(\lambda)+}(k)\,b^{(\lambda)+}(k)\,e^{ik\cdot x} + v^{(\lambda)+}(k)\,d^{(\lambda)}(k)\,e^{-ik\cdot x}\right]$$

$$j^0(x) = \int\frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\frac{m}{\omega_k}\frac{m}{\omega_{k'}}\sum_{\lambda,\lambda'}\Big[u^{(\lambda)+}(k)\,u^{(\lambda')}(k')\,b^{(\lambda)+}(k)\,b^{(\lambda')}(k')\,e^{ix\cdot(k-k')}$$
$$+ v^{(\lambda)+}(k)\,v^{(\lambda')}(k')\,d^{(\lambda)}(k)\,d^{(\lambda')+}(k') \times e^{ix\cdot(k'-k)}$$
$$+ (\text{stuff})\left[b^{(\lambda)}(k)\,d^{(\lambda')}(k) + \text{etc}\right]\Big]$$

Hence, $Q = \int j^0(x)\, d^3x$

$$= \int d^3x \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \frac{m}{\omega_k} \frac{m}{\omega_{k'}}$$

$$\times \sum_{\lambda,\lambda'} \Big[u^{(\lambda)+}(k) u^{(\lambda')}(k')\, b^{(\lambda)+}(k)\, b^{(\lambda')}(k')\, e^{ix\cdot(k-k')} + v^{(\lambda)+}(k) v^{(\lambda')}(k')\, d^{(\lambda)}(k)\, d^{(\lambda')+}(k')\, e^{ix\cdot(k'-k)} \Big]$$

But $\int \frac{d^3x}{(2\pi)^3} e^{i\vec{x}\cdot(\vec{k}-\vec{k}')} = \delta^3(\vec{k}-\vec{k}')$ so

$$Q = \int \frac{d^3k}{(2\pi)^3} d^3k' \frac{m}{\omega_k} \frac{m}{\omega_{k'}} \sum_{\lambda,\lambda'} \Big[u^{(\lambda)+}(k) u^{(\lambda')}(k')\, b^{(\lambda)+}(k)\, b^{(\lambda')}(k') + v^{(\lambda)+}(k) v^{(\lambda')}(k')\, d^{(\lambda)}(k)\, d^{(\lambda')+}(k') \Big] \delta^3(\vec{k}-\vec{k}')$$

Integrating oved d^3k' gives:

$$Q = \int \frac{d^3k}{(2\pi)^3} \left(\frac{m}{\omega_k}\right)^2 \sum_{\lambda,\lambda'} \Big[u^{(\lambda)+}(k) u^{(\lambda')}(k)\, b^{(\lambda)+}(k)\, b^{(\lambda')}(k) + v^{(\lambda)+}(k) v^{(\lambda')}(k)\, d^{(\lambda)}(k)\, d^{(\lambda)+}(k) \Big]$$

But $u^{(\lambda)+}(k) u^{(\lambda')}(k) = \frac{\omega_k}{m} \delta^{\lambda\lambda'}$

and $v^{(\lambda)+}(k) v^{(\lambda')}(k) = \frac{\omega_k}{m} \delta^{\lambda\lambda'}$

So $Q = \int \frac{d^3k}{(2\pi)^3} \frac{m}{\omega_k} \sum_{\lambda} \Big[b^{(\lambda)+}(k)\, b^{(\lambda)}(k) + d^{(\lambda)}(k)\, d^{(\lambda)+}(k) \Big]$

Next, since $\{ d^{(\lambda)}(k), d^{(\lambda)+}(k) \} = 1$ this is

$$Q = \int \frac{d^3k}{(2\pi)^3} \frac{m}{\omega_k} \sum_{\lambda} \Big[b^{(\lambda)+}(k)\, b^{(\lambda)}(k) - d^{(\lambda)+}(k)\, d^{(\lambda)}(k) \Big] + Q_0$$

(c)

Consider the states $b^{(\lambda)\dagger}(k)|0\rangle$ and $d^{(\lambda)\dagger}(k)|0\rangle$.

These are eigenstates of Q :

$$Q\, b^{(\lambda)\dagger}(k)\,|0\rangle = \int \frac{d^3k'}{(2\pi)^3}\frac{m}{\omega_{k'}} \sum_{\lambda'} b^{(\lambda')\dagger}(k')\, b^{(\lambda')}(k')\, b^{(\lambda)\dagger}(k)\,|0\rangle$$

But $\{ b^{(\lambda')}(k'),\, b^{(\lambda)\dagger}(k) \} = (2\pi)^3 \cdot \frac{\omega_k}{m}\, \delta^3(\vec{k}-\vec{k}')\,\delta^{\lambda\lambda'}$

So $Q\, b^{(\lambda)\dagger}(k)|0\rangle =$

$$= \int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{k'}} \sum_{\lambda'} \cdot (2\pi)^3 \frac{\omega_k}{m}\, \delta^3(\vec{k}-\vec{k}')\,\delta^{\lambda\lambda'}\, b^{(\lambda')\dagger}(k')|0\rangle$$

$$= b^{(\lambda)\dagger}(k)|0\rangle \qquad (\text{Eigenvalue is } +1)$$

$$Q\, d^{(\lambda)\dagger}(k)\,|0\rangle = \int \frac{d^3k'}{(2\pi)^3}\frac{m}{\omega_{k'}} \sum_{\lambda'} \left(-d^{(\lambda')\dagger}(k')\, d^{(\lambda')}(k')\right) d^{(\lambda)\dagger}(k)|0\rangle$$

But $\{ d^{(\lambda')}(k'),\, d^{(\lambda)\dagger}(k) \} = (2\pi)^3 \frac{\omega_k}{m}\, \delta^3(\vec{k}-\vec{k}')\,\delta^{\lambda\lambda'}$

So $Q\, d^{(\lambda)\dagger}(k)|0\rangle =$

$$= -\int \frac{d^3k}{(2\pi)^3}\frac{m}{\omega_{k'}} \sum_{\lambda'} (2\pi)^3 \frac{\omega_k}{m}\, \delta^3(\vec{k}-\vec{k}')\,\delta^{\lambda\lambda'}\, d^{(\lambda')\dagger}(k')|0\rangle$$

$$= -\, d^{(\lambda)\dagger}(k)|0\rangle \qquad (\text{Eigenvalue is } -1).$$

2. In the chiral representation,

$$\frac{1}{2}\vec{\Sigma}\cdot\hat{k} = \frac{1}{2|\vec{k}|}\begin{pmatrix} \vec{\sigma}\cdot\vec{k} & 0 \\ 0 & \vec{\sigma}\cdot\vec{k} \end{pmatrix}$$

(a) In the limit $E \gg m$,

$$u(k) = \frac{1}{\sqrt{2m(E+m)}}\begin{pmatrix} E+m+\vec{\sigma}\cdot\vec{k} & 0 \\ 0 & E+m-\vec{\sigma}\cdot\vec{k} \end{pmatrix} u(0)$$

$$\approx \sqrt{\frac{E}{2m}}\begin{pmatrix} 1+\vec{\sigma}\cdot\hat{k} & 0 \\ 0 & 1-\vec{\sigma}\cdot\hat{k} \end{pmatrix} u(0)$$

Thus, $\frac{1}{2}\vec{\Sigma}\cdot\hat{k}\, u(k) = \frac{1}{2}\sqrt{\frac{E}{2m}}\begin{pmatrix} \vec{\sigma}\cdot\hat{k}(1+\vec{\sigma}\cdot\hat{k}) & 0 \\ 0 & \vec{\sigma}\cdot\hat{k}(1-\vec{\sigma}\cdot\hat{k}) \end{pmatrix} u(0)$

$$= \frac{1}{2}\sqrt{\frac{E}{2m}}\begin{pmatrix} 1+\vec{\sigma}\cdot\hat{k} & 0 \\ 0 & -1+\vec{\sigma}\cdot\hat{k} \end{pmatrix} u(0)$$

$$= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} u(k)$$

$$= \frac{1}{2}\gamma^5 u(k) .$$

(b) $P_\pm = \frac{1}{2}(1 \pm \gamma^5)$

$$P_+ = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$P_- = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, $P_+ u(k) = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} E+m+\vec{\sigma}\cdot\vec{k} & 0 \\ 0 & 0 \end{pmatrix} u(0)$

which transforms like $e^{\vec{\sigma}\cdot\vec{\phi}/2}$

and $P_- u(k) = \frac{1}{\sqrt{2m(E+m)}} \begin{pmatrix} 0 & 0 \\ 0 & E+m-\vec{\sigma}\cdot\vec{k} \end{pmatrix} u(0)$

which transforms like $e^{-\vec{\sigma}\cdot\vec{\phi}/2}$.

(c) $\frac{1}{2}\vec{\Sigma}\cdot\hat{k}\, P_+ u(k) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u(k)$

$= +\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} u(k)$

$= +\frac{1}{2} P_+ u(k)$.

$\frac{1}{2}\vec{\Sigma}\cdot\hat{k}\, P_- u(k) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} u(k)$

$= -\frac{1}{2}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} u(k)$

$= -\frac{1}{2} P_- u(k)$.