Assignment #2

1. Consider the Lagrangian density

$$\mathcal{L} = (\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi$$

(a) The canonical energy-momentum tensor is given by

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} \partial^\nu \phi^* - g^{\mu\nu} \mathcal{L}$$

$$= (\partial^\mu \phi^*)(\partial^\nu \phi) + (\partial^\mu \phi)(\partial^\nu \phi^*) - g^{\mu\nu}\left((\partial^\rho \phi^*)(\partial_\rho \phi) - m^2 \phi^* \phi \right)$$

(b) The fields are represented as follows:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2\omega_k} \left(\alpha(k) e^{-ik\cdot x} + \beta^\dagger(k) e^{ik\cdot x} \right)$$

$$\phi^*(x) = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2\omega_k} \left(\beta(k) e^{-ik\cdot x} + \alpha^\dagger(k) e^{ik\cdot x} \right)$$

Derivatives with respect to x^μ are as follows:

$$\partial^\mu \phi(x) = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2\omega_k} \left((-ik^\mu)\alpha(k) e^{-ik\cdot x} + (ik^\mu)\beta^\dagger(k) e^{ik\cdot x} \right)$$

$$\partial^\mu \phi^*(x) = \int \frac{d^3k'}{(2\pi)^3} \cdot \frac{1}{2\omega_{k'}} \left((-ik'^\mu)\beta(k') e^{-ik'\cdot x} + (ik'^\mu)\alpha^\dagger(k') e^{ik'\cdot x} \right)$$

So $T^{\mu\nu} = \int \frac{d^3k}{(2\pi)^3}\cdot\frac{d^3k'}{(2\pi)^3}\cdot\frac{1}{\omega_k}\cdot\frac{1}{\omega_{k'}} \Big\{ (\text{stuff})\big(\alpha(k)\beta(k)+\beta^\dagger(k)\alpha^\dagger(k)\big)$

$+ k^\mu k'^\nu \alpha(k)\alpha^\dagger(k') e^{ix\cdot(k'-k)} + k^\mu k'^\nu \beta^\dagger(k)\beta(k') e^{ix\cdot(k-k')}$

$+ k'^\mu k^\nu \beta(k')\beta^\dagger(k) e^{ix\cdot(k-k')} + k'^\mu k^\nu \alpha^\dagger(k')\alpha(k) e^{ix\cdot(k'-k)}$

$- g^{\mu\nu}(\text{ more stuff }) \,. \Big\}$

The "charge" corresponding to the i^{th} component of $T^{\mu\nu}$ is

$$P^i = \int d^3x\, T^{0i} = \int d^3x \frac{d^3k}{(2\pi)^3}\cdot\frac{d^3k'}{(2\pi)^3}\cdot\frac{1}{2\omega_k}\cdot\frac{1}{2\omega_{k'}} \Big\{$$

$$\Big(\omega_k\cdot k'^i \alpha(k)\alpha^\dagger(k') e^{ix\cdot(k'-k)} + \omega_k\cdot k'^i \beta^\dagger(k)\beta(k') e^{ix\cdot(k-k')}$$

$$+ \omega_{k'} k^i \beta(k')\beta^\dagger(k) e^{ix\cdot(k-k')} + \omega_{k'} k^i \alpha^\dagger(k')\alpha(k) e^{ix\cdot(k'-k)}$$

Now, $\int \frac{d^3x}{(2\pi)^3} e^{i\vec{x}\cdot(\vec{k}-\vec{k}')} = \delta^3(\vec{k}-\vec{k}')$ so this is,

$$P^i = \int \frac{d^3k}{(2\pi)^3}\cdot\frac{1}{2\omega_k}\cdot\frac{1}{2\omega_{k'}} \Big(\omega_k k'^i \alpha(k)\alpha^\dagger(k') + \omega_k k'^i \beta^\dagger(k)\beta(k')$$

$$+ \omega_{k'}\cdot k^i \beta(k')\beta^\dagger(k) + \omega_{k'}\cdot k^i \alpha^\dagger(k')\alpha(k)\Big)$$

$$\times\, \delta^3(\vec{k}-\vec{k}')\, d^3k'$$

$$= \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\cdot\frac{1}{2\omega_k}\cdot k^i \Big(\alpha(k)\alpha^\dagger(k') + \alpha^\dagger(k)\alpha(k)$$

$$+ \beta^\dagger(k)\beta(k) + \beta(k)\beta^\dagger(k)\Big)$$

Next, using $[\alpha(k), \alpha^+(k)] = 1$ this can be written in normal order:

$$P^i = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2\omega_k} \cdot k^i \Big(2\alpha^+(k)\alpha(k) + 1 + 2\beta^+(k)\beta(k) + 1 \Big)$$

$$= \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2\omega_k} k^i \Big(\alpha^+(k)\alpha(k) + \beta^+(k)\beta(k) + 1 \Big)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} k^i \Big(N_\alpha(k) + N_\beta(k) + 1 \Big)$$

However, since the integrand is an odd function, the integral of the last term in the brackets is zero.

Therefore, $P^i = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2\omega_k} k^i \Big(N_\alpha(k) + N_\beta(k) \Big)$