Assignment # 1.

1. Show that the normalization of states:

$$\langle k|k'\rangle = (2\pi)^3 \cdot 2E \cdot \delta^3(\vec{k}-\vec{k}')$$

is invariant under a Lorentz boost in the x-direction.

The boost transforms E and k_x as follows:

$$\begin{aligned}
E &\rightarrow \hat{E} = \gamma E + \gamma\beta k_x \\
k_x &\rightarrow \hat{k}_x = \gamma k_x + \gamma\beta E \\
k_y &\rightarrow \hat{k}_y = k_y \\
k_z &\rightarrow \hat{k}_z = k_z
\end{aligned}$$

Then, $2E\,\delta^3(\vec{k}-\vec{k}') \rightarrow 2(\gamma E + \gamma\beta k_x)\,\delta(\gamma(k_x - k_x') + \gamma\beta(E-E'))$
$\cdot\, \delta(k_y - k_y')\,\delta(k_z - k_z')$

Let $u = k_x - k_x' \rightarrow k_x = u + k_x'$
then $\delta(f(u)) = \dfrac{\delta(u)}{f'(u)}$

In this case, $f(u) = \gamma(k_x - k_x') + \gamma\beta(E - E')$
$= \gamma(k_x - k_x') + \gamma\beta\left(\sqrt{|\vec{k}|^2 + m^2} - \sqrt{|\vec{k}'|^2 + m^2}\right)$

$$f'(u) = \frac{\partial f}{\partial k_x}\frac{\partial k_x}{\partial u} = \frac{\partial f}{\partial k_x} = \gamma + \frac{\gamma\beta k_x}{E} = \frac{\gamma E + \gamma\beta k_x}{E}$$

Thus, $2E \cdot \delta^3(\vec{k}-\vec{k}') \rightarrow 2(\gamma E + \gamma\beta k_x) \cdot \dfrac{E}{\gamma E + \gamma\beta k_x}\, \delta^3(\vec{k}-\vec{k}')$

$= 2E\,\delta^3(\vec{k}-\vec{k}')$.

Hence, this normalization is Lorentz invariant.

2. Show that the Lorentz invariant measure

$$\frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2E}$$

can also be written in the manifestly Lorentz covariant form,

$$\frac{d^4k}{(2\pi)^4} (2\pi)\, \delta(k^2 - m^2)\, \theta(k_0)$$

where $\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x < 0 \end{cases}$.

Consider $\int \frac{d^4k}{(2\pi)^4} (2\pi)\, \delta(k^2 - m^2)\, \theta(k_0)$

$$= \int_{k^0>0} \frac{d^4k}{(2\pi)^4} (2\pi)\, \delta(k^2 - m^2)$$

$$= \int_{k^0>0} \frac{d^4k}{(2\pi)^4} (2\pi)\, \delta(k_0^2 - |\vec{k}|^2 - m^2)$$

$$= \int_{k^0>0} \frac{d^4k}{(2\pi)^4} (2\pi)\, \delta\left(\left(k_0 - \sqrt{|\vec{k}|^2+m^2}\right)\left(k_0 + \sqrt{|\vec{k}|^2+m^2}\right)\right)$$

Next, recall the following property of delta functions with polynomial arguments:

$$\delta((x-a)(x-b)) = \frac{\delta(x-a)}{a-b} + \frac{\delta(x-b)}{b-a}$$

Hence, $\int \frac{d^4k}{(2\pi)^4} (2\pi)\, \delta(k^2 - m^2)\, \theta(k_0)$

$$= \int_{k_0>0} \frac{d^4k}{(2\pi)^4} (2\pi)\, \frac{\delta\left(k_0 - \sqrt{|\vec{k}|^2+m^2}\right)}{k_0 + \sqrt{|\vec{k}|^2+m^2}} + \int_{k_0>0} \frac{d^4k}{(2\pi)^4} (2\pi)\, \frac{\delta\left(k_0 + \sqrt{|\vec{k}|^2+m^2}\right)}{k_0 - \sqrt{|\vec{k}|^2+m^2}}$$

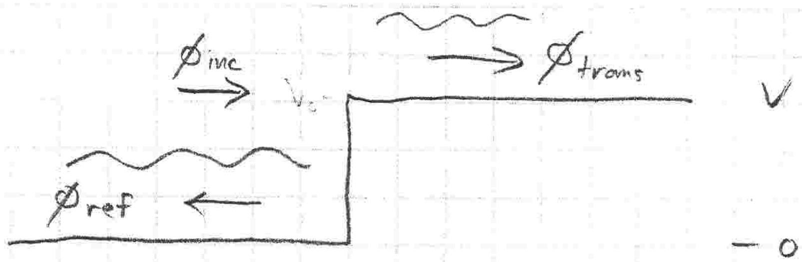
The argument of the second delta function never vanishes since we only integrate over positive k_0. Furthermore, the delta function dominates the pole at $k_0 = \sqrt{|\vec{k}|^2+m^2}$ so the second integral must vanish.

Integrating the first integral over k_0 then gives:

$$\int \frac{d^4k}{(2\pi)^4} \cdot (2\pi)\,\delta(k^2-m^2)\,\theta(k_0) = \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2\sqrt{|\vec{k}|^2+m^2}}$$

$$= \int \frac{d^3k}{(2\pi)^3} \cdot \frac{1}{2E} \,.$$

3.



The incident wave is $\psi_I(x,t) = e^{-i(Et-kx)}$ and the reflected and transmitted waves are

$$\psi_R(x,t) = a e^{-i(Et+kx)}$$

$$\psi_T(x,t) = b e^{-i(E't-k'x)}$$

where $E' = E - V$.

(a) If $E = \frac{k^2}{2m}$, then $k' = \sqrt{2mE'}$

$$= \sqrt{2m(E-V)}.$$

At $x=0$ we also have to match the wavefunctions:

$$(1+a)e^{-iEt} = be^{-iE't}$$

$$1+a = be^{i(E-E')t} = be^{iVt}$$

and the derivatives:

$$ik(1-a) = k' b e^{i(E-E')t}$$

$$1-a = \frac{k'}{k} b e^{i(E-E')t} = \frac{k'}{k} b e^{iVt}$$

Hence, $\quad b\left(1+\frac{k'}{k}\right)e^{iVt} = 2$

$$b = \frac{2ke^{-iVt}}{k+k'}$$

$$a = be^{iVt} - 1$$

$$= \frac{2k-(k+k')}{k+k'} = \frac{k-k'}{k+k'}$$

(b) When $E < V$, $k' = \sqrt{2m(E-V)} = i\sqrt{2m(V-E)}$

Hence, the transmitted wavefunction is of the form

$$\psi_T(x,t) = \frac{2ke^{-iVt}}{k+i\sqrt{2m(V-E)}} e^{-iE't} e^{-x\sqrt{2m(V-E)}}$$

$$= \frac{2ke^{iEt}}{k+i\sqrt{2m(V-E)}} e^{-x/\lambda}$$

where $\lambda = \frac{1}{\sqrt{2m(V-E)}} > 0$.

Hence, when $V \gg E$, the wave dies off quickly for $x \gg \lambda$!

(c) When $E = \sqrt{k^2 + m^2}$ and $E' = E - V$ we have

$$k' = \sqrt{(E-V)^2 - m^2}$$

but the relations for a and b remain unchanged:

$$a = \frac{k-k'}{k+k'}$$

$$b = \frac{2ke^{-iVt}}{k+k'}$$

(d) A freely propagating transmitted wave corresponds to a real valued k'.

This can will occur when $(E-V)^2 > m^2$

but this can happen if either $E > V+m$
or $E < V-m$.

(e) $E > V+m$ corresponds to the normal classical transmission process.

However, the freely propagating waves with $E' < V-m$ are classically forbidden since they have energy less than the potential in which they propagate.

These could be interpreted as an antiparticle or the absence of a state in a sea of states with $E < V-m$.