Particle's with spin ½.

Schrödinger's equation is first order in $\frac{\partial}{\partial t}$ and did not produce negative energy solutions:

$$\frac{p^2}{2m}\psi = E\psi \Rightarrow \frac{-\hbar^2}{2m}\nabla^2\psi = i\hbar\frac{\partial}{\partial t}\psi$$

The Klein-Gordon equation was second order in $\frac{\partial}{\partial t}$ and did produce negative energy solutions.

Can we construct a first order equation that satisfies $H^2\psi = (p^2+m^2)\psi = E^2\psi$?

$$H\psi = i\frac{\partial}{\partial t}\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi = E\psi$$

$$H^2\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)^2\psi = E^2\psi = (|\vec{p}|^2 + m^2)\psi.$$

$$(\vec{\alpha}\cdot\vec{p} + \beta m)^2 = (\alpha_i p_i + \beta m)(\alpha_j p_j + \beta m)$$

$$= \alpha_i\alpha_j p_i p_j + \alpha_i\beta m p_i + \beta\alpha_j m p_j + \beta^2 m^2$$

$$= |\vec{p}|^2 + m^2$$

So $\alpha_i\alpha_i = 1$ $\quad \alpha_i\alpha_j + \alpha_j\alpha_i = 0$ when $i \neq j$

$\beta^2 = 1$ $\quad \alpha_i\beta + \beta\alpha_i = 0$

Clearly, α_i, β are not real numbers.

We can construct them using matrices.

Pauli matrices were encountered when describing spin ½ particles:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

They satisfy $\sigma_i \sigma_j = i \varepsilon_{ijk} \sigma_k + \delta_{ij}$.

That is,

$$\sigma_1 \sigma_1 = I$$
$$\sigma_2 \sigma_2 = I$$
$$\sigma_3 \sigma_3 = I$$

Sometimes we write $\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

$$\sigma_1 \sigma_2 = i \sigma_3$$
$$\sigma_1 \sigma_3 = -i \sigma_2$$
$$\sigma_2 \sigma_3 = i \sigma_1$$

etc.

The Pauli matrices anticommute:

$$\sigma_1 \sigma_2 + \sigma_2 \sigma_1 = 0$$

or in general, $\sigma_i \sigma_j + \sigma_j \sigma_i = 2 \delta_{ij}$.

We can construct $\vec{\alpha}$, β from $\sigma_0, \vec{\sigma}$:

Let $\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ ← 4×4 matrices.

These satisfy the desired commutation relations.

If we multiply everything by β then we can write this in covariant form:

$$i\beta \frac{\partial}{\partial t} \psi = \left(\beta \vec{\alpha} \cdot \vec{p} + \beta^2 m \right) \psi$$

$$= \left(-i\beta \vec{\alpha} \cdot \nabla + \cancelto{1}{\beta^2} m \right) \psi$$

or, $\left(i\beta \frac{\partial}{\partial t} + i\beta\vec{\alpha} \cdot \nabla - m \right) \psi = 0$

If we write $\gamma^0 = \beta$

$\vec{\gamma} = \beta \vec{\alpha}$

then $\left(i \gamma^\mu \partial_\mu - m \right) \psi = 0$

This is the Dirac equation.

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

More notation, We multiply 4-vectors by γ-matrices so frequently that we will write $\gamma^\mu \partial_\mu = \not{\partial}$ or in general for any 4-vector a^μ, $\gamma^\mu a_\mu = \not{a}$.

The γ-matrices satisfy many identities.
[see handout].

For a complex wavefunction, ϕ we could take the complex conjugate ϕ^*.

Now, ψ is a 1×4 column matrix so we will work with something similar.

called a Dirac spinor

Take the Hermetian conjugate:

$$[(i\gamma^\mu\partial_\mu - m)\psi]^\dagger = \psi^\dagger(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_\mu - m)$$

$$= \psi^\dagger(-i\gamma^0\gamma^\mu\gamma^0\overleftarrow{\partial}_\mu - \gamma^0\gamma^0 m)$$

$$= \bar{\psi}(-i\gamma^\mu\gamma^0\overleftarrow{\partial}_\mu - \gamma^0 m)$$

$$= -\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m)\gamma^0 = 0$$

So the adjoint spinor satisfies

$$\bar{\psi}(i\gamma^\mu\overleftarrow{\partial}_\mu + m) = 0.$$

The continuity equation reads:

$$\bar{\psi}(i\overleftarrow{\not{\partial}} - m)\psi + \bar{\psi}(i\overrightarrow{\not{\partial}} + m)\psi = 0$$

$$i\bar{\psi}\overleftarrow{\not{\partial}}\psi + i\bar{\psi}\overrightarrow{\not{\partial}}\psi = i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0.$$

The electron current density is identified with

$$j^\mu = -e\bar{\psi}\gamma^\mu\psi.$$

Physics 564 - Pauli and Dirac Matrices

The Pauli matrices are

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1}$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2}$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

They satisfy the following identity:

$$\sigma_i \sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij} \tag{4}$$

where

$$\epsilon_{ijk} = \begin{cases} 1 & \text{for even permutations of } (ijk) \\ -1 & \text{for odd permutations of } (ijk) \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

It follows that

$$(\vec{\sigma}\cdot\vec{a})(\vec{\sigma}\cdot\vec{b}) = (\sigma_i a_i)(\sigma_j b_j) \tag{6}$$

$$= i a_i b_j \epsilon_{ijk}\sigma_k + a_i b_j \delta_{ij} \tag{7}$$

$$= i(\vec{a}\times\vec{b})\cdot\vec{\sigma} + \vec{a}\cdot\vec{b} \tag{8}$$

$$= i(\vec{\sigma}\times\vec{a})\cdot\vec{b} + \vec{a}\cdot\vec{b} \tag{9}$$

We will use the following representation for the Dirac matrices:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}. \tag{11}$$

They satisfy

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu \tag{12}$$

$$= 2g^{\mu\nu} \tag{13}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{14}$$

The tensor $\sigma^{\mu\nu}$ is defined:

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) \tag{15}$$

The matrix γ^5 is defined

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{16}$$

and satisfies the identities

$$\begin{aligned} \left\{\gamma^\mu, \gamma^5\right\} &= \gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0 && (17) \\ \gamma^{5\dagger} &= \gamma^5 && (18) \\ (\gamma^5)^2 &= 1 && (19) \end{aligned}$$

Solutions to Dirac's equation

Returning to the first form:

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\,\psi(x) = E\,\psi(x)$$

we write $\psi(x) = u(\vec{p})\,e^{-ip\cdot x}$ where $u(\vec{p})$ can be expressed

$$u(\vec{p}) = \begin{pmatrix} u_A(\vec{p}) \\ u_B(\vec{p}) \end{pmatrix}$$

where u_A and u_B are 1×2 column vectors.

$$(\vec{\alpha}\cdot\vec{p} + \beta m)\,\psi(x) = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}\begin{pmatrix} u_A(\vec{p}) \\ u_B(\vec{p}) \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} u_A(\vec{p}) \\ u_B(\vec{p}) \end{pmatrix}$$

So $$\begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} E-m & 0 \\ 0 & E+m \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

Which gives two coupled equations:

$$\vec{\sigma}\cdot\vec{p}\,u_B = (E-m)\,u_A$$
$$\vec{\sigma}\cdot\vec{p}\,u_A = (E+m)\,u_B$$

or $$u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m}\,u_B$$

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,u_A$$

Suppose $E > 0$. Then we can pick $u_A = \chi^{(s)}$
where $\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then $u^{(s)}(\vec{p}) = N \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi^{(s)} \end{pmatrix}$ describes the two spin states of an electron.

But we can still have $E < 0$ ... In this case,

Let $u^{(s+2)}(\vec{p}) = N \begin{pmatrix} \frac{-\vec{\sigma} \cdot \vec{p}}{|E|+m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$

If p^μ is the physical 4-momentum of a positron then we can define

$$\underbrace{v^{(s)}(p) = u^{(5-s)}}(-p) = N \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

ie, $v^{(1)}(p) = u^{(4)}(-p)$
$v^{(2)}(p) = u^{(3)}(-p)$

The v spinors are introduced for convenience when describing positrons. They satisfy a slightly modified form of the Dirac equation:

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$\psi(x) = u(p) e^{-ip \cdot x}$$

$(\not{p} - m) u(p) = 0$ for particles

$(-\not{p} - m) u(-p) = 0$ for antiparticles

But $v(p) = u(-p)$ so the v-spinors satisfy

$$(-\not{p} - m)\, v(p) = 0$$

or

$$(\not{p} + m)\, v(p) = 0 .$$

Normalization

Previously, we required that

$$\int \rho \, dV = \int \phi^* \phi \, dV = \frac{2E}{V}$$

and then set $V = 1$ since it always cancelled when calculating $d\sigma$ or $d\Gamma$.

In the case of spinors we require that

$$\int \psi^\dagger \psi \, dV = 2E$$

$$\vec{\sigma}^\dagger = \vec{\sigma}$$

But

$$\psi^\dagger \psi = u^\dagger u = |N|^2 \left(\chi^{(s)T} , \frac{\chi^{(s)T} \vec{\sigma} \cdot \vec{p}}{E+m} \right) \begin{pmatrix} \chi^{(s)} \\ \dfrac{\vec{\sigma} \cdot \vec{p}}{E+m} \chi^{(s)} \end{pmatrix}$$

$$= |N|^2 \left(1 + \frac{\chi^{(s)T} (\vec{\sigma} \cdot \vec{p})(\vec{\sigma} \cdot \vec{p}) \chi^{(s)}}{(E+m)^2} \right)$$

$$= |N|^2 \left(1 + \frac{|\vec{p}|^2}{(E+m)^2} \right)$$

$$= |N|^2 \left(1 + \frac{E^2 - m^2}{(E+m)^2} \right)$$

$$= |N|^2 \left(1 + \frac{(E-m)}{(E+m)} \right)$$

$$= |N|^2 \left(\frac{E + m + E - m}{E + m} \right)$$

$$= |N|^2 \cdot \frac{2E}{E+m} \equiv 2E$$

Hence, $N = \sqrt{E+m}$

So the exact forms of u and v are:

$$u^{(s)}(\vec{p}) = \begin{pmatrix} \sqrt{E+m}\, \chi^{(s)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\, \chi^{(s)} \end{pmatrix} \qquad E = \sqrt{|\vec{p}|^2 + m^2}$$

$$v^{(s)}(\vec{p}) = \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\, \chi'^{(s)} \\ \sqrt{E+m}\, \chi'^{(s)} \end{pmatrix} \qquad E = \sqrt{|\vec{p}|^2 + m^2}$$

Where $\chi'^{(s)} = \chi^{(3-s)}$

ie, $\chi'^{(1)} = \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\chi'^{(2)} = \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Feynman rules:

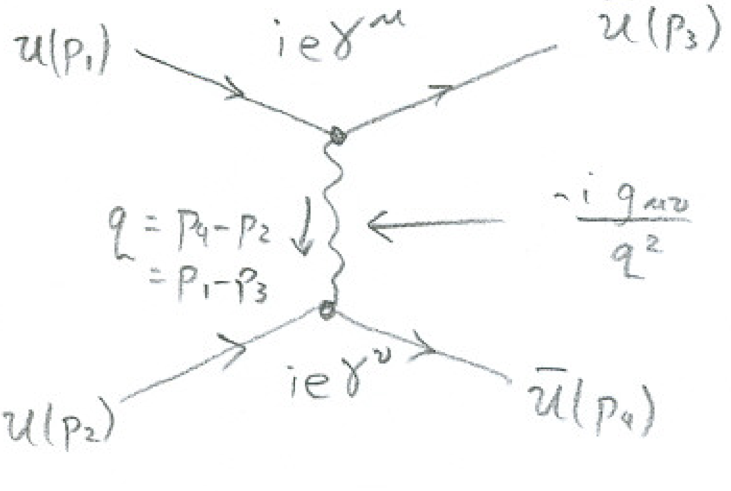
The electromagnetic current is

$$j_\mu^{fi}(x) = -e\bar{\psi}_f \gamma_\mu \psi_i$$
$$= -e\bar{u}_f \gamma_\mu u_i \, e^{i(p_f - p_i)\cdot x}$$

For spinless fields this was

$$j_\mu^{fi}(x) = -e(p_{f\mu} + p_{i\mu})e^{i(p_f - p_i)\cdot x}$$

So the Feynman rules for electrons and positrons are:



$$-i\mathcal{M} = (ie\bar{u}_3\gamma^\mu u_1)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(ie\bar{u}_4\gamma^\nu u_2)$$

where $u_1 = u^{(s_1)}(p_1)$
$\bar{u}_3 = \bar{u}^{(s_3)}(p_3)$
etc.