13 Oct 2005

We are trying to calculate matrix elements for spinless charged particles in an electromagnetic field.

If we return to the point where we derived Fermi's golden rule, we had an expression

$$C_n(t) = \frac{-i}{\hbar} \frac{\langle n|V|i\rangle}{i(E_n-E_i)/\hbar - \Gamma/2} e^{i(E_n-E_i)t'/\hbar - \Gamma t'/2} \Big|_0^t$$

We can absorb the $e^{i(E_n-E_i)t'/\hbar}$ into the matrix element to give

$$C_n(t) = \frac{-i}{\hbar} \frac{\langle n,t'|V|i,t'\rangle}{i(E_n-E_i)/\hbar - \Gamma/2} e^{-\Gamma t'/2} \Big|_0^t$$

This doesn't change anything since we found that the denominator ensured energy conservation when we sum $|c_n(t)|^2$ over all final states.

We found that a charged particle in an electromagnetic field was described by the Hamiltonian

$$[\underbrace{(\partial^\mu \partial_\mu + m^2)}_{\text{Free Hamiltonian}} + \underbrace{V}_{\text{Perturbation due to EM field.}}]\phi(x) = 0$$

where $V = -ie(\partial^\mu A_\mu + A^\mu \partial_\mu)$

We need to evaluate the transition amplitude:

$$T_{fi} = -i \langle f | V | i \rangle \int e^{i(E_f - E_i)t}\, dt$$

$$= -i \int d^3x' d^3x \, \langle f | x' \rangle \langle x' | V | x \rangle \langle x | i \rangle \, e^{i(E_f - E_i)t}\, dt$$

If, as is often the case, V is local, depending only on the fields at point x, then

$$\langle x' | V | x \rangle = \delta^3(\vec{x}' - \vec{x}) \langle x | V | x \rangle$$

and

$$T_{fi} = -i \int d^3x \, \phi_f^*(\vec{x}) \langle x | V | x \rangle \phi_i(\vec{x}) \, e^{i(E_f - E_i)t}\, dt$$

$$= -i \int d^4x \, \phi_f^*(x) V(x) \phi_i(x)$$

For the charged particles in an electromagnetic field this is

$$T_{fi} = -i \int d^4x \, \phi_f^*(x) \left(-ie \, \partial^\mu A_\mu + A^\mu \partial_\mu \right) \phi_i(x)$$

But $\int d^4x \, \phi_f^*(x) \, \partial^\mu \left(A_\mu \phi_i(x) \right)$

$$= \cancelto{0}{\int d^4x \, \partial^\mu \left(\phi_f^*(x) A_\mu \phi_i(x) \right)} - \int d^4x \left(\partial^\mu \phi_f^*(x) \right) A_\mu \phi_i(x)$$

So $$T_{fi} = -e \int d^4x \left(\phi_f^*(x) \partial^\mu \phi_i(x) - \left[\partial^\mu \phi_f(x) \right] \phi_i(x) \right) A_\mu(x)$$

$$= -i \int j_{fi}^\mu(x) A_\mu(x) \, d^4x$$

Where $j^{\mu}_{fi}(x)$ is the transition current:

$$j^{\mu}_{fi}(x) = -ie\left(\phi_f^{*}\,\partial^{\mu}\phi_i - (\partial^{\mu}\phi_f^{*})\,\phi_i\right)$$

between initial and final states.

If $\phi_i(x) = e^{-ik_i\cdot x}$ $\quad \phi_f^{*}(x) = e^{ik_f\cdot x}$ then

$$j^{\mu}_{fi}(x) = -e\,(k_i^{\mu} + k_f^{\mu})\,e^{i(k_f-k_i)\cdot x}$$

$$T_{fi} = -ie\int d^4x\,(k_i^{\mu}+k_f^{\mu})\,A_{\mu}(x)\,e^{i(k_f-k_i)\cdot x}$$

$$= ie\,(k_f^{\mu}+k_i^{\mu})\,A_{\mu}(q)$$

where $q^{\mu} = k_f^{\mu} - k_i^{\mu}$ and $A_{\mu}(q)$ is the Fourier transform of $A_{\mu}(x)$.

For two particles scattering from eachother $A^\mu(x)$ is the field created by the other particle.

Maxwell's equations:

$$\partial^\mu \partial_\mu A^\nu(x) = j'^{\nu}_{fi}(x)$$

Where $j'^{\nu}_{fi}(x) = -e(p_f^\nu + p_i^\nu)\, e^{i(p_f - p_i)\cdot x}$

We can write $A^\nu(x) = \frac{1}{(2\pi)^4}\int d^4q\, e^{iq\cdot x} A^\nu(q)$

Then $$\partial^\mu \partial_\mu A^\nu(x) = \frac{1}{(2\pi)^4}\int d^4q\,(-q^2)\, e^{iq\cdot x} A^\nu(q)$$

$$= \frac{1}{(2\pi)^4}\int d^4q\, e^{iq\cdot x} j'^{\nu}_{fi}(q)$$

So $A^\nu(q) = \left(\frac{-1}{q^2}\right) j'^{\nu}_{fi}(q)$

$$j'^{\nu}_{fi}(q) = \int d^4x\, e^{-iq\cdot x} j'^{\nu}_{fi}(x)$$

$$= -e(p_f^\nu + p_i^\nu)\int d^4x\, e^{-iq\cdot x} e^{i(p_f - p_i)\cdot x}$$

$$= -e(p_f^\nu + p_i^\nu)\cdot(2\pi)^4\, \delta^4(p_f - p_i - q)$$

So $$A^\nu(x) = -e(p_f^\nu + p_i^\nu)\int d^4q \left(\frac{-1}{q^2}\right) e^{iq\cdot x}\, \delta^4(p_f - p_i - q)$$

$$= -e(p_f^\nu + p_i^\nu)\cdot\left(\frac{-1}{q^2}\right) e^{i(p_f - p_i)\cdot x}$$

where $q = p_f - p_i$

$$\text{So} \quad T_{fi} = -i \int d^4x \, j^{\mu}_{fi}(x) \, A_{\mu}(x)$$

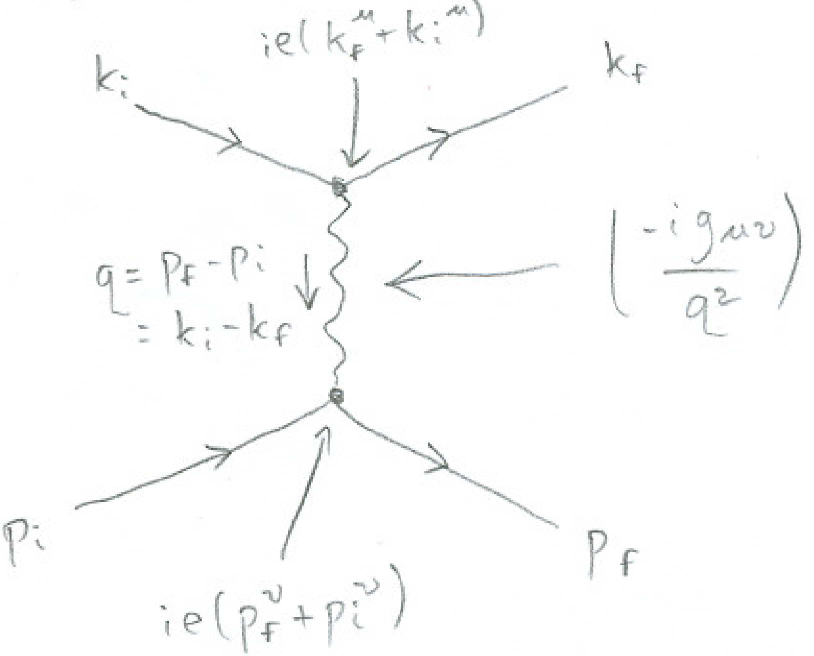
$$= -i \left(e(k_f^{\mu} + k_i^{\mu})\right) \left(\frac{-g_{\mu\nu}}{q^2}\right) \left(e(p_f^{\nu} + p_i^{\nu})\right) \int d^4x \, e^{i(k_f - k_i)\cdot x} \, e^{i(p_f - p_i)\cdot x}$$

$$= -i \, (2\pi)^4 \, \delta^4(p_f + k_f - p_i - k_i) \, \mathcal{M}$$

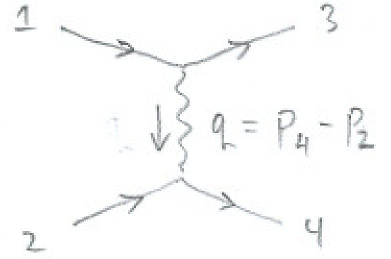
$$\text{Where } -i\mathcal{M} = \left(ie(k_f^{\mu} + k_i^{\mu})\right) \left(-\frac{i g_{\mu\nu}}{q^2}\right) \left(ie(p_f^{\nu} + p_i^{\nu})\right)$$

is called the "Reduced Matrix Element".

Rather than go through this algebra each time, we can write down some rules from which it can be constructed:



If we relabel the particles :



then $$d\sigma = \frac{|\mathcal{M}|^2}{F} dQ$$

where $$F = 4\left((p_1 \cdot p_2)^2 - m_1^2 m_2^2 \right)^{1/2}$$

$$dQ = (2\pi)^4 \delta^4 (p_3 + p_4 - p_1 - p_2) \frac{d^3 p_3}{(2\pi)^3 \cdot 2E_3} \frac{d^3 p_4}{(2\pi)^3 \cdot 2E_4}$$

Let's evaluate this ...

$$-i\mathcal{M} = \left(ie(p_3^\mu + p_1^\mu)\right)\left(-\frac{i g_{\mu\nu}}{q^2}\right)\left(ie(p_4^\nu + p_2^\nu)\right)$$

$$= ie^2 \frac{(p_3 + p_1)\cdot(p_4 + p_2)}{q^2}$$

$$= ie^2 \left(\frac{p_3 \cdot p_4 + p_3 \cdot p_2 + p_1 \cdot p_4 + p_1 \cdot p_2}{q^2} \right)$$

We can write these in terms of the Mandlestam variables:

$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2$$

$$t = (p_3 - p_1)^2 = (p_4 - p_2)^2$$

$$u = (p_3 - p_2)^2 = (p_4 - p_1)^2$$

$$p_3 \cdot p_4 = \frac{1}{2}(s - p_3^2 - p_4^2)$$

$$p_3 \cdot p_2 = -\frac{1}{2}(u - p_3^2 - p_2^2)$$

$$p_1 \cdot p_4 = -\frac{1}{2}(u - p_1^2 - p_4^2)$$

$$p_1 \cdot p_2 = \frac{1}{2}(s - p_1^2 - p_2^2)$$

So $(p_3 + p_1) \cdot (p_4 + p_2) = s - u$

$$q^2 = (p_4 - p_2)^2 = t$$

So $-i\mathcal{M} = ie^2 \left(\frac{s-u}{t}\right)$

Suppose we evaluate the cross section in the lab frame where particle 2 is initially at rest. $p_2 = (M, \vec{0})$

$$p_1 = (E_1, \vec{p}_1)$$

The flux factor is then

$$\begin{aligned} F &= 4((p_1 \cdot p_2)^2 - M^2 m^2)^{1/2} \\ &= 4((E_1 M)^2 - M^2 m^2)^{1/2} \\ &= 4M(E_1^2 - m^2)^{1/2} \\ &= 4M|\vec{p}_1| \end{aligned}$$

$$dQ = (2\pi)^4 \delta^4(p_3 + p_4 - p_1 - p_2) \frac{d^3p_3}{(2\pi)^3 2E_3} \frac{d^3p_4}{(2\pi)^3 2E_4}$$

$$= \frac{(2\pi)}{2E_4} \delta(E_3 + E_4 - E_1 - E_2) \frac{d^3p_3}{(2\pi)^3 \cdot 2E_3}$$

where now $\vec{p}_4 = \vec{p}_1 + \vec{p}_2 - \vec{p}_3$. this also fixes E_4 in terms of p_1, p_2, p_3.

$$= \frac{(2\pi)}{2E_4} \delta(E_3 + E_4 - E_1 - E_2) \frac{|\vec{p}_3|^2 dp_3 \, d\Omega}{(2\pi)^3 \cdot 2E_3}$$

But $dE = d(|\vec{p}_3|^2 + m_3^2)^{1/2} = \dfrac{|\vec{p}_3| \, dp_3}{E_3}$

So $$dQ = \frac{(2\pi)}{2E_4} \delta(E_3 + E_4 - E_1 - E_2) \frac{|\vec{p}_3| \, dE_3 \, d\Omega}{(2\pi)^3 \cdot 2}$$

$$= \frac{|\vec{p}_3| \, d\Omega}{4E_4 (2\pi)^2}$$

So $$d\sigma = \frac{|\vec{p}_3|}{16E_4 M |\vec{p}_1|} \cdot \frac{e^4}{(2\pi)^2} \cdot \left(\frac{s-u}{t}\right)^2 d\Omega$$

It is common to write $\alpha = \dfrac{e^2}{4\pi}$

So $$\frac{d\sigma}{d\Omega} = \frac{|\vec{p}_3|}{4|\vec{p}_1| E_4 M} \alpha^2 \left(\frac{s-u}{t}\right)^2$$

$$s = (p_1 + p_2)^2 = m^2 + M^2 + 2p_1 \cdot p_2 = m^2 + M^2 + 2E_1 M$$

$$t = (p_3 - p_1)^2 = 2m^2 - 2p_1 \cdot p_3 = -2(E_1 E_3 - |\vec{p}_1||\vec{p}_3| \cos\theta)$$

$$u = (p_3 - p_2)^2 = m^2 + M^2 - 2p_2 \cdot p_3 = m^2 + M^2 - 2E_3 M$$

$$\frac{d\sigma}{d\Omega} = \frac{|\vec{p}_3|}{4|\vec{p}_1|E_4 M}\alpha^2\left(\frac{2M(E_1+E_3)}{-2(E_1E_3-|\vec{p}_1||\vec{p}_3|\cos\theta)}\right)^2$$

Examine the limiting case where $M \to \infty$.
Then $E_4 \simeq M$, $E_3 = E_1 = E$, $|\vec{p}_3| = |\vec{p}_1| = p$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4M^2}\alpha^2\left(\frac{-2ME}{E^2-p^2\cos\theta}\right)^2$$

$$= \frac{\alpha^2E^2}{(m^2+p^2(1-\cos\theta))^2}$$

In the non-relativistic limit, $E \to m$, $m^2 \gg p^2$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2m^2}{p^4(1-\cos\theta)^2} = \frac{\alpha^2m^2}{4p^4\sin^4\theta/2}$$

But $\frac{p^2}{2m} = K$ so $\frac{p^4}{4m^2} = K^2$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16K^2\sin^4\theta/2}$$

This is the Rutherford scattering formula

(eg. Sakurai, Chapter 7)

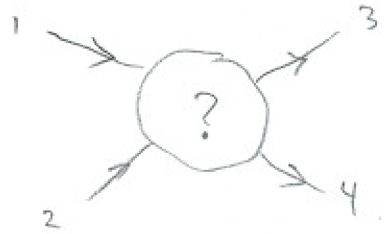
Also in the High energy limit,

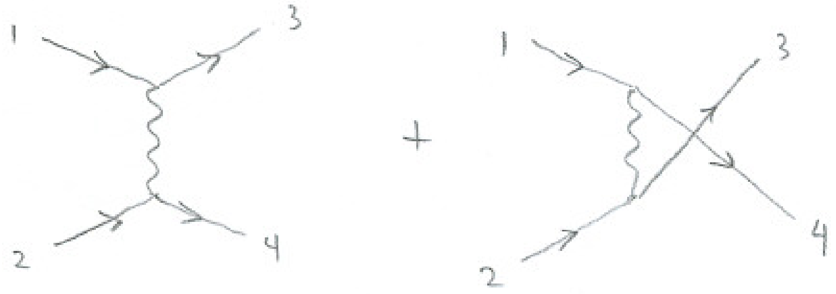
$$p \to E \quad , \quad E \gg m ,$$

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E^2 \sin^4 \theta/2}$$

In this example it was assumed that the particles were distinguishable. For example, if they were π^+ and π^-.

If they were identical then there are two diagrams to consider:





The second diagram is the same as the first one but with $3 \leftrightarrow 4$.

Hence
$$s = (p_1 + p_2)^2 = (p_3 + p_4)^2 \longrightarrow s$$
$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2 \longrightarrow (p_1 - p_4)^2 = u$$
$$u = (p_1 - p_4)^2 = (p_2 - p_3)^2 \longrightarrow (p_1 - p_3)^2 = t$$

So the reduced matrix element for scattering identical spinless particles is

$$-i\mathcal{M} = ie^2\left(\frac{s-u}{t}\right) + ie^2\left(\frac{s-t}{u}\right)$$