Purdue University Department of Physics

Physics 56400
Introduction to Elementary Particle Physics I

Lecture 4

Fall 2019 Semester

Prof. Matthew Jones

Special Relativity

- 4-momentum of a particle at rest:

$$p = \left(m, \vec{0}\right)$$

- Lorentz transformation for a boost along the x-axis:

$$L = \begin{pmatrix} \gamma & \gamma\beta & & 0 \\ \gamma\beta & \gamma & & \\ & 0 & 1 & \\ & & & 1 \end{pmatrix}$$

- 4-momentum for a particle moving with velocity β:

$$p' = (E', \vec{p}\,')$$

$$\left.\begin{aligned} E' &= \gamma m \\ p'_x &= \gamma\beta m \\ p'_y = p'_z &= 0 \end{aligned}\right\} \quad \begin{aligned} \gamma &= E'/m \\ \beta &= p'/E' \\ \gamma\beta &= p'/m \end{aligned}$$

Special Relativity

- Inner products of 4-vectors:

$$a \cdot b = a_\mu b^\mu = a^0 b^0 - \vec{a} \cdot \vec{b}$$

- In particular,

$$p^2 = p \cdot p = E^2 - |\vec{p}|^2 = m^2$$

  - It should be clear from context that this is *not* referring to the y-component of the 3-momentum.
- For massless particles (ie, photons and neutrinos):

$$E = |\vec{p}|$$

Particle Decays

- Particle A with mass M_A is at rest.

$$p_A = \left(M_A, \vec{0}\right)$$

- Suppose it decays into two particles, a and b, with masses m_a and m_b.
- What are the energies and momenta of particles a and b?
- Energy-momentum conservation:

$$p_A = p_a + p_b$$
$$p_A^2 = M_A^2 = (p_a + p_b)^2$$
$$= p_a^2 + 2\, p_a \cdot p_b + p_b^2$$
$$m_a^2 + 2\, p_a \cdot p_b + m_b^2$$

Particle Decays

- But, $p_b = p_A - p_a$ so

$$p_a \cdot p_b = p_a \cdot (p_A - p_a)$$
$$= p_a \cdot p_A - p_a^2$$
$$= E_a M_A - m_a^2$$

- Hence,

$$M_A^2 = m_a^2 + 2(E_a M_A - m_a^2) + m_b^2$$
$$= 2\, M_A E_a + m_b^2 - m_a^2$$

- Therefore,

$$\boxed{E_a = \frac{M_A^2 + m_a^2 - m_b^2}{2M_A}}$$

Particle Decays

$$E_a = \frac{M_A^2 + m_a^2 - m_b^2}{2M_A}$$
$$E_b = \frac{M_A^2 + m_b^2 - m_a^2}{2M_A}$$

- Observations:
  - If $m_a = m_b$, then $E_a = E_b = M_A/2$.
  - If $M_A \gg m_a, m_b$ then $E_a = E_b = M_A/2$.
  - Don’t try this on massless particles ($M_A = 0$)…

Particle Decays

- The momenta can be calculated using

$$|\vec{p}| = \sqrt{E^2 - m^2}$$

- Because particle A was initially at rest,

$$\vec{p}_a = -\vec{p}_b$$

- The magnitude of the momentum of either particle is

$$\boxed{|\vec{p}_a| = |\vec{p}_b| = \frac{\left[(M_A^2 - (m_a + m_b)^2)(M_A^2 - (m_a - m_b)^2)\right]^{1/2}}{2\, M_A}}$$

- What if the decaying particle was not at rest?

Particle Decays

- If particle A was not at rest, then we can calculate the Lorentz boost parameters:

$$p_A = (E_A, \vec{p}_A)$$

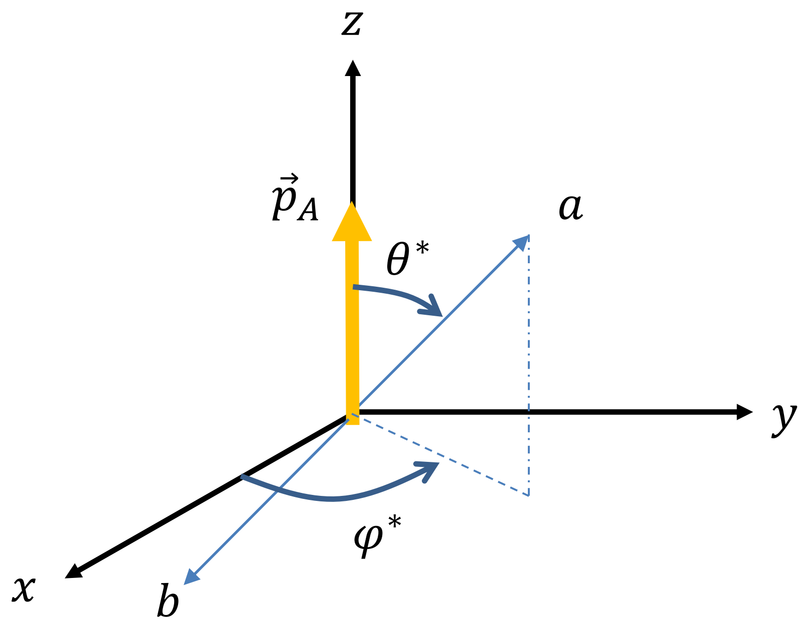
$$\gamma = \frac{E_A}{M_A}$$

$$\beta = \frac{|\vec{p}_A|}{E_A}$$

$$\gamma\beta = \frac{|\vec{p}_A|}{M_A}$$

Particle Decays

- Sometimes we denote quantities that are defined in a particular rest frame with asterisks…
- We can define the z-axis to be in the direction of $\vec{p}_A$
- Then, suppose that particle a is emitted with polar angles (θ^*, φ^*) in the rest frame of particle A:



Particle Decays

- In the rest frame of particle A,

$$p_{ax}^{*} = |\vec{p}^{*}| \sin\theta^{*} \cos\varphi^{*}$$
$$p_{ay}^{*} = |\vec{p}^{*}| \sin\theta^{*} \sin\varphi^{*}$$
$$p_{az}^{*} = |\vec{p}^{*}| \cos\theta^{*}$$

- Remember that p_{ax}^{*} and p_{ay}^{*} are not affected by a boost along the z-axis.
- In the lab frame,

$$\varphi = \varphi^{*}$$

Particle Decays

- In the lab frame,

$$E_a = \gamma E_a^* + \gamma\beta p_{az}^* = \gamma E_a^* + \gamma\beta|\vec{p}^*|\cos\theta^*$$
$$p_{az} = \gamma\beta E_a^* + \gamma p_{az}^* = \gamma\beta E_a^* + \gamma|\vec{p}^*|\cos\theta^*$$

- Recall that we know what γ and β are, so we are (in principle) finished.
- These are not necessarily tidy expressions, but they can be calculated in this way. Computers will do it without complaining.
- In practice, this is the most straight forward way to calculate the properties of decay products.

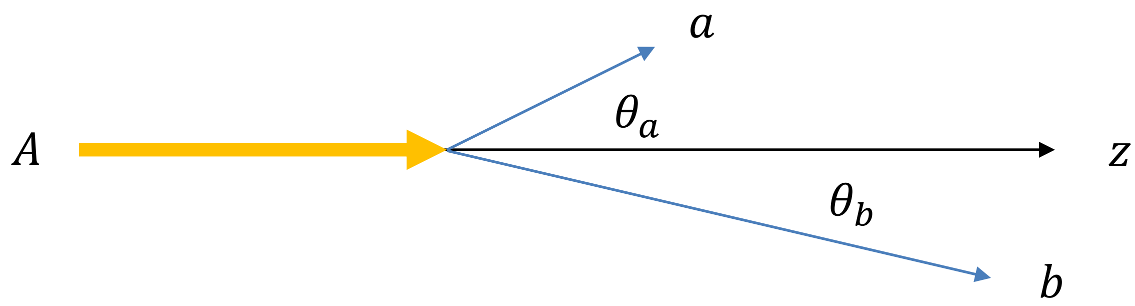
Particle Decays

- The transverse components (the components perpendicular to the z-axis) are particularly interesting because we defined the boost direction to be along the z-axis.
- From momentum conservation, the transverse component of the momenta are equal and opposite:

$$p_T = \sqrt{p_{ax}^2 + p_{ay}^2} = \sqrt{p_{bx}^2 + p_{by}^2}$$

- Next, we can calculate the angles, with respect to the z-axis, that a and b are emitted in the lab frame.

Particle Decays



$$\tan\theta_a = \frac{p_T}{p_{az}}$$
$$\tan\theta_b = \frac{p_T}{p_{bz}}$$

- But recall that θ^* was defined by the direction of particle a.

$$E_b = \gamma E_b^* - \gamma\beta|\vec{p}^*|\cos\theta^*$$
$$p_{bz} = \gamma\beta E_b^* - \gamma|\vec{p}^*|\cos\theta^*$$

Examples

- An ${}^{241}_{95}Am$ nucleus decays into ${}^{237}_{93}Np$ and an α-particle.

$$M_A = (241.056829\ \mathrm{u})\ (931.5\ \mathrm{MeV/u})$$
$$m_a = (237.048173\ \mathrm{u})(931.5\ \mathrm{MeV/u})$$
$$m_b = (4.002603\ \mathrm{u})(931.5\ \mathrm{MeV/u})$$
$$E_b = \frac{M_A^2 + m_b^2 - m_a^2}{2M_A}$$
$$= (4.00856\ \mathrm{u})(931.5\ \mathrm{MeV/u})$$
$$= 3734\ \mathrm{MeV}$$

Examples

- Next, we can calculate the momentum of the α-particle:

$$|p| = \sqrt{E^2 - m^2}$$

$$|\vec{p}_b| = (931.5\ \mathrm{MeV/u})\sqrt{(4.00856\ \mathrm{u})^2 - (4.002603\ \mathrm{u})^2}$$

$$= 203.5\ \mathrm{MeV}$$

$$\beta = \frac{p}{E} = \frac{203.5\ \mathrm{MeV}}{3734\ \mathrm{MeV}} = 0.054$$

- Recall that when we analyzed this problem using non-relativistic kinematics we obtained $\beta = 0.055$.

Collisions

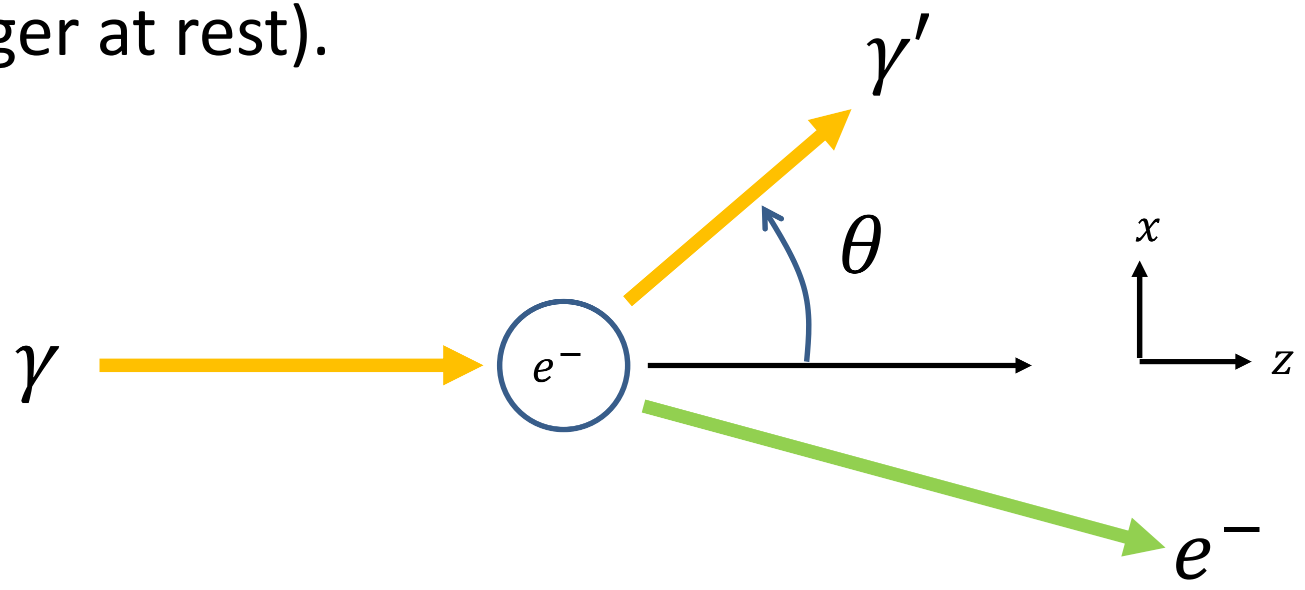
- Consider the collision of two particles:

$$A + B \to a + b + \cdots$$

- Again, we can apply energy-momentum conservation to derive constraints on the momenta of the final state particles.
- Some simple cases:
  - Fixed target case, $|\vec{p}_B| = 0$
  - Center-of-mass frame: $\vec{p}_A + \vec{p}_B = 0$

Example: Compton scattering

- A gamma ray scatters off a stationary electron.
- The initial state is the photon and the electron.
- The final state is a scattered photon (with different energy and direction) and a scattered electron (no longer at rest).



Example: Compton scattering

- Pick the z-axis to be in the direction of the incident γ.
- Pick the x-axis to lie in the scattering plane.
- Initial 4-momenta:

$$p_\gamma = (E, 0,0, E)$$
$$p_e = (m_e, 0,0,0)$$

- Final 4-momenta:

$$p'_\gamma = (E', E' \sin\theta\,, 0, E' \cos\theta)$$
$$p'_e = p_\gamma + p_e - p'_\gamma$$
$$= (E + m_e - E', -E' \sin\theta\,, 0, E - E' \cos\theta)$$

Example: Compton scattering

- We would like to derive a relation between the final state photon energy and its scattering angle.

$$|\vec{p}_e'|^2 = {E_e'}^2 - m_e^2$$
$$= (\vec{p}_\gamma - \vec{p}_\gamma') \cdot (\vec{p}_\gamma - \vec{p}_\gamma')$$
$$E^2 + E'^2 - 2EE' \cos\theta$$

- But the final state electron energy can be written:

$$E_e'^2 = (E + m_e - E')^2$$

(lots of stuff cancels)

$$(E - E')m_e = EE'(1 - \cos\theta)$$
$$1 - \cos\theta = \left(\frac{1}{E'} - \frac{1}{E}\right)m_e$$

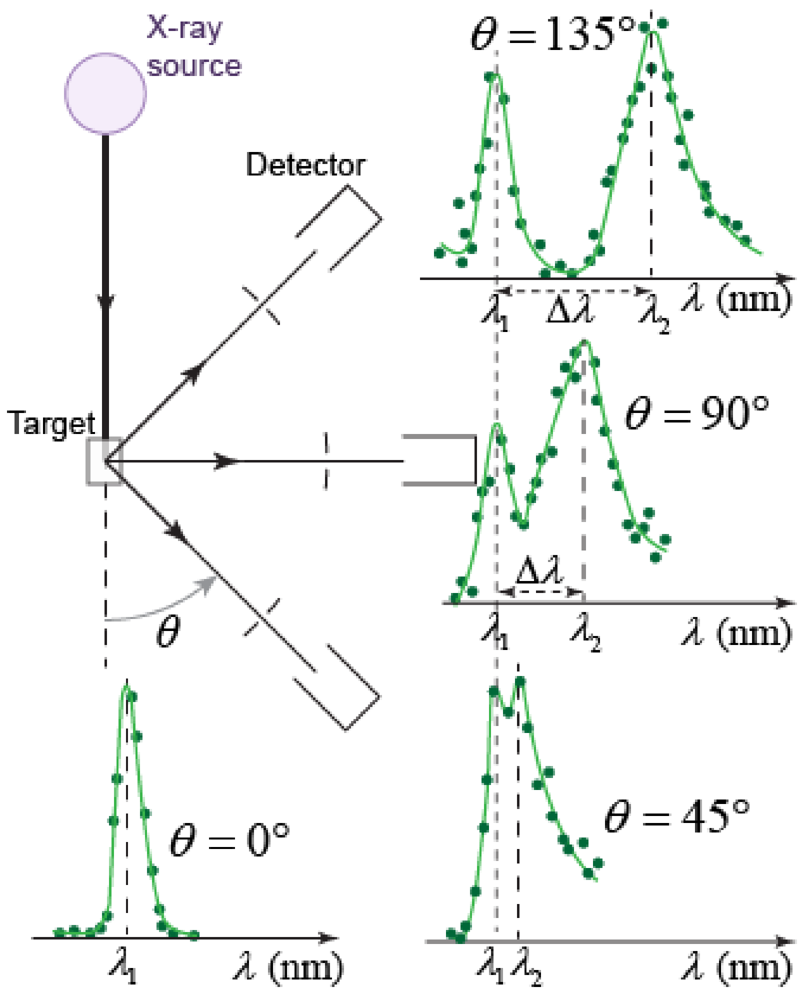
Example: Compton scattering

$$\frac{1}{E'} - \frac{1}{E} = \frac{1 - \cos\theta}{m_e}$$

- Recall that $E = hc/\lambda$
- The change in wavelength can be written:

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta) = \frac{2\pi\hbar}{m_e c}(1 - \cos\theta)$$

Example: Compton Scattering



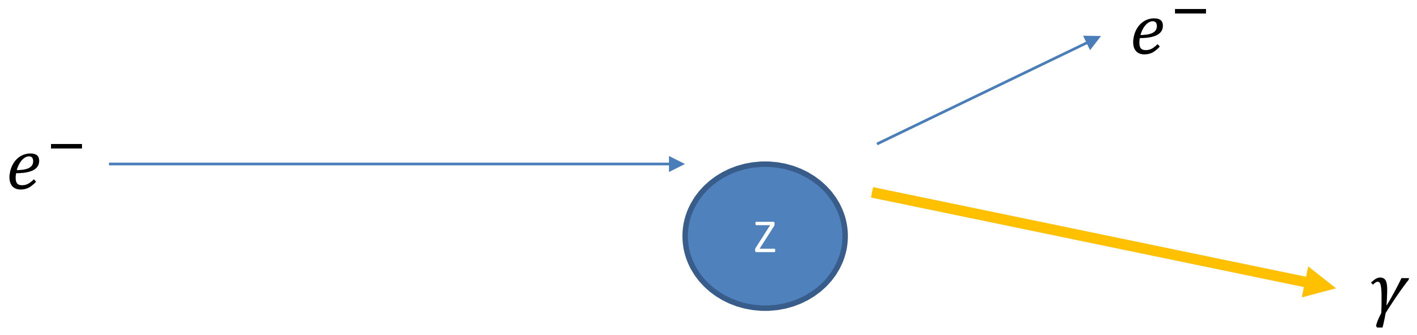
We have only used energy-momentum conservation and special relativity to analyze the kinematics.

So far, we can say nothing about the rate at which scattering occurs.

For that, we need to apply relativistic quantum mechanics (Quantum Electrodynamics).

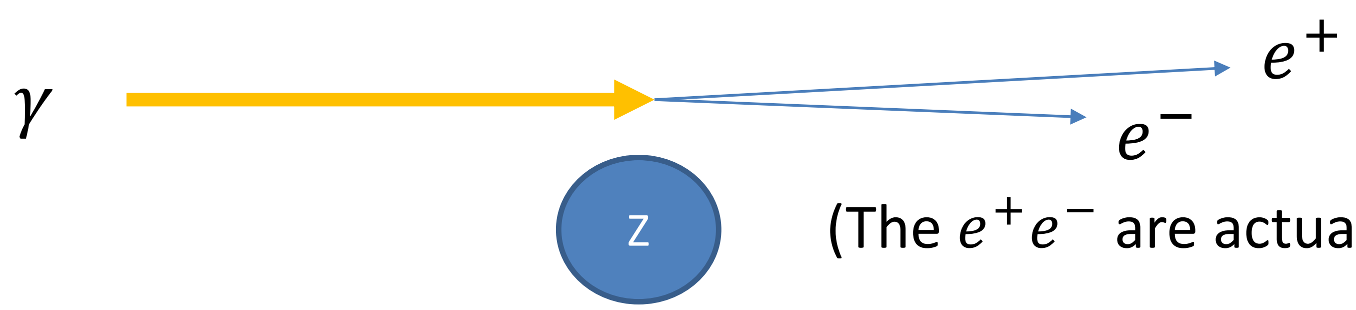
Example: Bremsstrahlung

- Accelerated electrons can radiate x-rays:



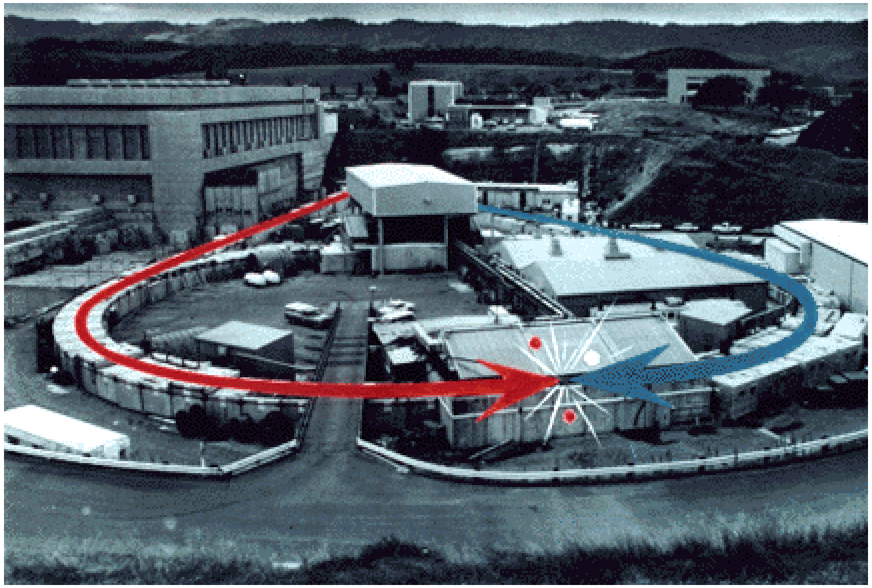
Example: Pair production

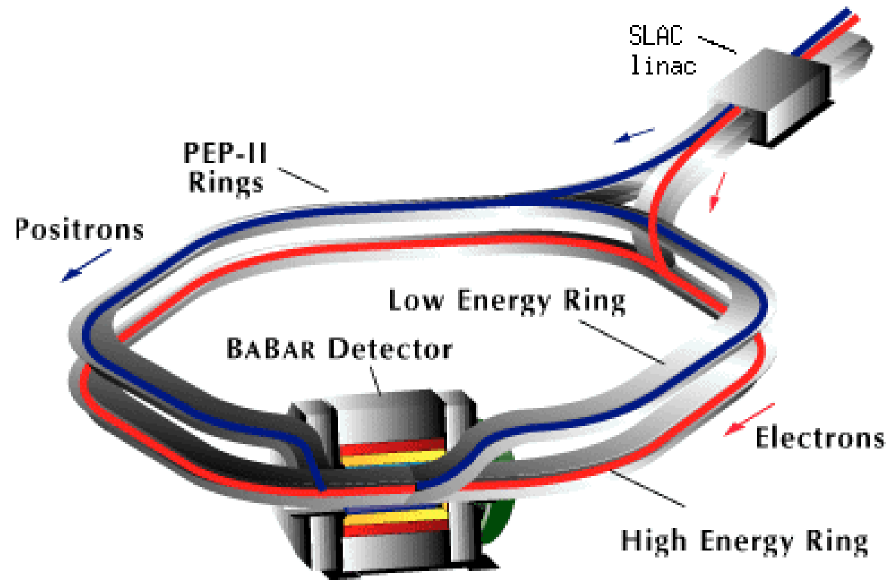
- We saw that a massless particle could not decay into a pair of massive particles.
- However, a massless particle could "collide" with another particle and convert into a pair of particles.
- The final state contains three particles.
  - To conserve electric charge or other quantum numbers, they must be particle/anti-particle pairs.



(The e^+e^- are actually co-linear)

Example: Colliding Beam Experiments





SPEAR storage ring:
e^+e^- collisions at a center-of-mass energy of 8 GeV
(each beam was 4 GeV).

PEP-II storage ring:
$9.0\ \text{GeV}$ electrons collide with $3.1\ \text{GeV}$ positrons.

Particle Lifetimes

- Unstable particles decay with a characteristic lifetime:

$$N(t) = N_0 e^{-t/\tau}$$

- The lifetime, τ, is a fundamental property of the particle.
- Sometimes it is called the “proper” lifetime to remind us that it is defined in the rest-frame of the particle.
- The observed time of decay, t, depends on the reference frame.

Particle Lifetimes

- Suppose a particle decays a time Δt after it was produced, measured in its rest frame.
- Suppose the particle has momentum $\vec{p}$ in the lab frame.
- Questions:
  - What is the time difference between production and decay in the lab frame?
  - How far does it travel in the lab frame before it decays?
- 4-vector for the proper decay time:

$$x = (\Delta t, \vec{0})$$

Particle Lifetimes

- Boost parameters (in terms of E and p) which are obviously defined in the lab frame):

$$\gamma = E/m$$
$$\beta = p/E$$
$$\gamma\beta = p/m$$

- Decay-time 4-vector in the lab frame:

$$x' = Lx$$
$$\Delta t' = \gamma \Delta t = \frac{E}{m} \Delta t$$
$$\Delta x' = \gamma\beta \Delta t = \frac{p}{m} \Delta t$$

Particle Lifetimes

- In normal units,

$$\Delta x' = \gamma\beta c \Delta t$$

- Example: cosmic rays are mostly muons which have a mass of $m = 105.7\ \mathrm{MeV}$ and a lifetime of $\tau = 2.2\ \mu s$.
- Suppose their typical momentum is $10\ \mathrm{GeV}$.
- What fraction has decayed after traveling 5 km?

$$\Delta t = \frac{\Delta x'}{\gamma\beta c} = \frac{m\Delta x'}{pc} = 0.176\ \mu s$$

- The fraction that decayed is

$$f = 1 - e^{-\Delta t/\tau} = 0.077$$

- Note that it takes them about $16.7\ \mu s$ to travel this distance (7.6 lifetimes).

Particle Lifetimes

- Example: Pions are also a component of cosmic rays at high altitudes.
- Pions mass $m = 139.6\ \mathrm{MeV}$ and lifetime $\tau = 26\ \mathrm{ns}$.
- Suppose their typical momentum was $10\ \mathrm{GeV}$.
- What fraction has decayed after traveling 5 km?

$$\Delta t = \frac{\Delta x'}{\gamma\beta c} = \frac{m\Delta x'}{pc} = 0.233\ \mu s$$

- The fraction that decayed is

$$f = 1 - e^{-\Delta t/\tau} = 0.9999$$