PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

# Physics 56400

# Introduction to Elementary Particle Physics I

Lecture 22

Fall 2019 Semester

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# Announcement about the Assignment

- It is important to have definition of couplings that is consistent with equations that use them
- Previously we had defined

$$c_L = \frac{1}{2}(c_V + c_A)$$
$$c_R = \frac{1}{2}(c_V - c_A)$$

- But in this case, the sum of $\gamma$- and $Z$-exchange amplitudes should actually be of the form:

$$|\mathcal{M}_{LL}|^2 \propto \left|1 + 4r c_L^e c_L^\mu\right|^2$$

(there is a factor of 4 that cancels the $(1/2)^2$ in $c_L^e c_L^\mu$)

# Announcement about the Assignment

- The notation used in some text books defines the coupling constants this way:

$$c_L = c_V + c_A$$
$$c_R = c_V - c_A$$

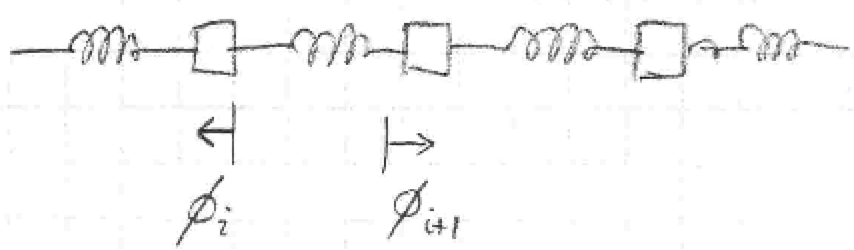
- In this case, the amplitudes are what was previously written in class:

$$|\mathcal{M}_{LL}|^2 \propto \left|1 + r c_L^e c_L^\mu\right|^2$$

- Sorry for any confusion this might have caused…

# Lagrangian Formulation

- A central problem in physics is determining how dynamical variables evolve with time
- In classical physics, dynamical variables might be the coordinates of a set of masses that exert forces on each other
- A useful example is a set of masses attached with springs:



# Lagrangian Formulation

- Total kinetic energy:

$$T = \sum_i \frac{1}{2} m \, \dot{\phi}_i^2$$

- Total potential energy:

$$V = \sum_i \frac{1}{2} k(\phi_{i+1} - \phi_i)^2$$

- Lagrangian:

$$L = T - V$$

# Lagrangian Formulation

- Lagrange's equations follow from the requirement that the action is stationary:

$$S = \int_{t_1}^{t_2} L \, dt$$

$$\delta S = 0$$

- Lagrange's equations:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}_i} - \frac{\partial L}{\partial \phi_i} = 0$$

# Lagrangian Formulation

- In the continuum limit, the discrete index $i$ can be replaced by a continuous variable, $x$.
- Then, $\phi(x)$ is the displacement from equilibrium of the particle located at position $x$.

$$L = \int_0^{\ell} \left( \frac{1}{2}\rho\dot{\phi}(x)^2 - \frac{1}{2}Y\left(\frac{d\phi}{dx}\right)^2 \right) dx$$

$$\rho = \frac{dm}{dx} \qquad Y = \frac{dk}{dx}$$

- The integrand is the Lagrangian density, $\mathcal{L}$

$$\frac{\partial\mathcal{L}}{\partial\phi} - \frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot{\phi}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\right) = 0$$

# Lagrangian Formulation

- Consider the infinite mass-spring system:

$$\mathcal{L} = \frac{1}{2}\rho\dot{\phi}(x)^2 - \frac{1}{2}Y\phi'(x)^2$$

- Lagrange’s equations:

$$-\rho\frac{\partial}{\partial t}\dot{\phi} + Y\frac{\partial}{\partial x}\phi' = 0$$

- This is the wave equation:

$$\frac{\partial^2\phi}{\partial t^2} - \frac{Y}{\rho}\frac{\partial^2\phi}{\partial x^2} = 0$$

# Quantum Field Theory

- In Quantum Field Theory, the fields $\phi(x)$ are operators.
- They can be expressed in terms of creation and annihilation operators that act on n-particle states.
- Lagrange's equations describe how these fields must evolve with time
  - Dirac equation (massive spin ½ fields)
  - Maxwell's equations (massless spin 1 fields)

# Quantum Field Theory

- In 4-dimensional space-time we can write

$$\delta S = \delta \int d^4x \, \mathcal{L}(\phi, \partial_\mu \phi) = 0$$

$$\partial_\mu \phi = \frac{\partial \phi}{\partial x^\mu}$$

- Lagrange's equations can then be written

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

# Charged Scalar Fields

- Most of the mathematics can be illustrated using a charged scalar field
- There are no internal degrees of freedom (unlike Dirac fermions or electromagnetic fields)

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu \phi^*)\left(\partial_\mu \phi\right) + \frac{1}{2}m^2 \phi^* \phi$$

- Equations of motion:

$$\left(\partial^\mu \partial_\mu - m^2\right)\phi(x) = 0$$
$$\left(\partial^\mu \partial_\mu - m^2\right)\phi^*(x) = 0$$

# Global Symmetries

- All physical observables will depend on the modulus-squared of scattering amplitudes
- Redefining the phase of the fields should not have an observable consequence

$$\phi \to \phi' = e^{-i\Lambda}\phi \approx (1 - i\Lambda)\phi$$
$$\phi^* \to \phi'^* = e^{i\Lambda}\phi^* \approx (1 + i\Lambda)\phi^*$$

- In this case, $\Lambda$ is an arbitrarily small, real constant

$$\delta\phi = (\phi' - \phi) = -i\Lambda\phi$$
$$\delta\phi^* = \left(\phi'^* - \phi^*\right) = i\Lambda\phi^*$$
$$\delta\left(\partial_\mu\phi\right) = -i\Lambda\partial_\mu\delta\phi$$
$$\delta\left(\partial_\mu\phi^*\right) = i\Lambda\partial_\mu\delta\phi^*$$

- So the change in the Lagrangian is

$$\delta\mathcal{L} = \frac{1}{2}i\Lambda(\partial^\mu\phi^*)\left(\partial_\mu\delta\phi\right) - \frac{1}{2}i\Lambda(\partial^\mu\delta\phi^*)\left(\partial_\mu\phi\right)$$
$$-\frac{1}{2}i\Lambda m^2\phi^*\phi + \frac{1}{2}i\Lambda m^2\phi^*\phi = 0$$

# Local Symmetries

- Next, consider the phase to be a continuous function of space and time:

$$\phi \to \phi' = e^{-i\lambda(x)}\phi \approx (1 - i\lambda(x))\phi$$
$$\phi^* \to \phi'^* = e^{i\lambda(x)}\phi^* \approx (1 + i\lambda(x))\phi^*$$
$$\delta\phi = (\phi' - \phi) = -i\lambda(x)\phi$$
$$\delta\phi^* = \left(\phi'^* - \phi^*\right) = i\lambda(x)\phi^*$$

- Derivatives:

$$\partial_\mu\phi \to \partial_\mu\phi' = \left(1 - i\lambda(x)\right)\partial_\mu\phi - i\left(\partial_\mu\lambda(x)\right)\phi$$
$$\delta\left(\partial_\mu\phi\right) = \partial_\mu\phi' - \partial_\mu\phi = -i\lambda(x)\partial_\mu\phi - i\left(\partial_\mu\lambda(x)\right)\phi$$

(Likewise for $\phi^*$)

# Local Symmetries

- The Lagrangian is not invariant under this local gauge transformation:

$$\delta\mathcal{L} = \frac{1}{2} i((\partial^\mu \phi^*)\left(\partial_\mu \lambda(x)\right)\phi - \left(\partial_\mu \lambda(x)\right)\phi^*(\partial^\mu \phi)$$

- What we really only care about is that the action is stationary

$$\delta S = \int d^4x\, \delta\mathcal{L}$$

- Integration by parts:

$$\delta S = (surface\ term)$$

$$-\frac{i}{2}\int d^4x\, \lambda(x)\partial_\mu((\partial^\mu \phi^*)\,\phi - \phi^*(\partial^\mu \phi))$$

- $\lambda(x)$ is arbitrary, so there must be a conserved current

# Local Symmetries

$$\partial_\mu\big((\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)\big) = 0$$

- This is of the form:

$$\partial_\mu J^\mu = 0$$

- Conserved current:

$$J^\mu = (\partial^\mu \phi^*)\phi - \phi^*(\partial^\mu \phi)$$

- This is Noether's theorem:

"Local symmetries imply the existence of conserved currents."

- But what if we wanted the Lagrangian to be invariant and not just the action?

# Local Symmetries

- Under a local gauge transformation we found that:

$$\mathcal{L} \rightarrow \mathcal{L}' = \mathcal{L} + \left(\partial^\mu \lambda(x)\right) J_\mu(x) + \mathcal{O}(\lambda^2)$$

- If we want this to be unchanged, then we can introduce a new field, $A^\mu$, that couples to $J_\mu$ so as to cancel the unwanted term.

$$\mathcal{L} = (\partial^\mu \phi^*)\left(\partial_\mu \phi\right) - m^2 \phi^* \phi - e A^\mu J_\mu$$

- The new field must transform as follows:

$$A^\mu \rightarrow A'^\mu = A^\mu + \frac{1}{e} \partial^\mu \lambda(x)$$

# Local Symmetries

- However, the Lagrangian is still not invariant:

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} - 2eA^{\mu}\left(\partial_{\mu}\lambda(x)\right)\phi^{*}\phi$$

- But, if we add one more term, the unwanted part can be cancelled.
- Gauge invariant Lagrangian:

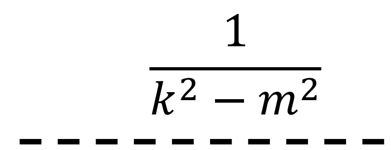
$$\mathcal{L} = (\partial^{\mu}\phi^{*})\left(\partial_{\mu}\phi\right) - m^{2}\phi^{*}\phi - eA^{\mu}J_{\mu} + e^{2}A_{\mu}A^{\mu}\phi^{*}\phi$$

- This is particularly useful in field theory because we can just read off the Feynman rules…

# Feynman Rules

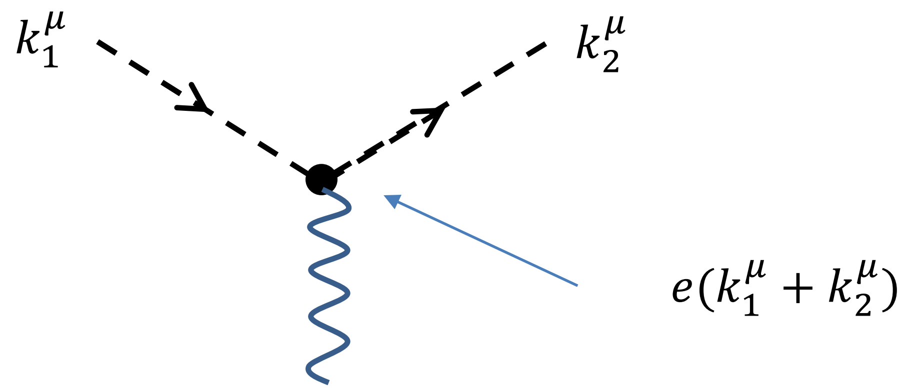
- Massive scalar propagator:

$$(\partial^\mu \phi^*)(\partial_\mu \phi) - m^2 \phi^* \phi$$




- Current coupling to the gauge field:

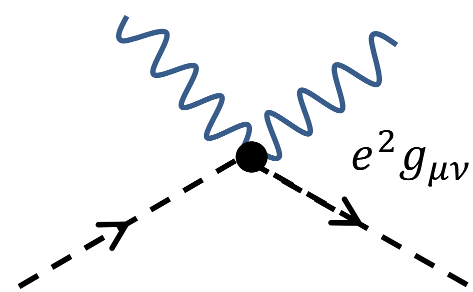
$$eA^\mu J_\mu$$




- Coupling to two gauge fields:

$$e^2 A_\mu A^\mu \phi^* \phi$$

(not present for Dirac fermions)



# Covariant Derivatives

- All the problems started with the derivative operators.
- The Lagrangian would be naturally invariant if the derivative operators transformed nicely:

$$D_\mu \phi \rightarrow D'_\mu \phi' = e^{-i\lambda(x)} D_\mu \phi$$
$$D^*_\mu \phi^* \rightarrow D^{*\prime}_\mu \phi'^* = e^{i\lambda(x)} D^*_\mu \phi^*$$

- The covariant derivative operators should be this:

$$D_\mu = \partial_\mu + ieA_\mu$$
$$D^*_\mu = \partial_\mu - ieA_\mu$$

- The Lagrangian is then invariant when written this way:

$$\mathcal{L} = \left(D^*_\mu \phi^*\right)(D^\mu \phi) - m\phi^* \phi$$

# Yang-Mills Gauge Symmetry

- The previous example was for a field that was invariant under a single phase transformation
- This is referred to as $U(1)$ symmetry
- Suppose we have a set of 3 fields that are invariant under rotations

$$\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} \rightarrow \begin{pmatrix}\phi_1{}'\\ \phi_2{}'\\ \phi_3{}'\end{pmatrix} = \begin{pmatrix}\cos\lambda_3(x) & \sin\lambda_3(x) & 0\\ -\sin\lambda_3(x) & \cos\lambda_3(x) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix}$$

- For very small rotations:

$$\vec{\phi} \rightarrow \vec{\phi}' = \vec{\phi} + \begin{pmatrix}0 & \lambda_3(x) & 0\\ -\lambda_3(x) & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\phi_1\\ \phi_2\\ \phi_3\end{pmatrix} + \mathcal{O}(\lambda^2)$$

# Yang-Mills Gauge Symmetry

- In general, these transformations work like a cross-product:

$$\vec{\phi} \to \vec{\phi}' = \vec{\phi} - \vec{\lambda}(x) \times \vec{\phi}$$

- We can also write this in abstract index notation:

$$\phi_i \to \phi_i' = \phi_i - \varepsilon_{ijk}\lambda_j(x)\phi_k$$

- The coefficients $\varepsilon_{ijk}$ are specific to the group of rotations, but other groups will have different coefficients:

$$\phi_i \to \phi_i' = \phi_i - f_{ijk}\lambda_j(x)\phi_k$$

- For now we will assume $SU(2)$ symmetry and use the cross products.

# Yang-Mills Gauge Symmetry

- If we want the Lagrangian to be locally gauge invariant, we need to introduce a set of gauge fields to construct the covariant derivatives:

$$D_\mu \vec{\phi} = \partial_\mu \vec{\phi} + g\overrightarrow{W_\mu} \times \vec{\phi}$$

- The gauge fields need to transform as follows:

$$\overrightarrow{W_\mu} \to \overrightarrow{W_\mu'} = \overrightarrow{W_\mu} - \vec{\lambda}(x) \times \vec{\phi} + \frac{1}{g}\partial_\mu \vec{\lambda}(x)$$

- Introduce a term like $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$:

$$\overrightarrow{W}^{\mu\nu} = \partial^\mu \overrightarrow{W}^\nu - \partial^\nu \overrightarrow{W}^\mu + g\overrightarrow{W}^\mu \times \overrightarrow{W}^\nu$$

- Then the Lagrangian is invariant when written:

$$\mathcal{L} = \left(D_\mu \vec{\phi}\right) \cdot \left(D_\mu \vec{\phi}\right) - m\vec{\phi} \cdot \vec{\phi} - \frac{1}{4}\overrightarrow{W}^{\mu\nu} \cdot \overrightarrow{W_{\mu\nu}}$$

# Fermions

- Lagrangians for Fermions are just constructed using the same covariant derivative operators:

$$\mathcal{L} = \bar{\psi}\left(i\gamma^{\mu} D_{\mu} + m\right)\psi$$

- The standard model groups fermions into left-handed doublets and right-handed singlets:

$$e_L = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \qquad e_R = (e^-)_R$$

- Weak interactions couple to left-handed Fermions and are invariant under SU(2) transformations (rotations in "weak isospin" space).

# Fermion Masses

- Explicit mass terms are not gauge invariant:

$$m_e \bar{u} u = m_e \bar{u}_R u_L + m_e \bar{u}_L u_R$$

- We can't make a gauge invariant combination out of

$$\bar{e}_L = \begin{pmatrix} \bar{\nu}_e \\ \overline{e^-} \end{pmatrix}_L \text{ and } e_R = (e^-)_R$$

- Furthermore, we can't make gauge invariant mass terms for the gauge bosons of the form

$$M\, \vec{W}_\mu \cdot \vec{W}^\mu$$

- Everything works very nicely if everything is massless.
- The trick is to add another field that has couplings to all other fields that "look" like mass terms.