Purdue University Department of Physics

Physics 56400

Introduction to Elementary Particle Physics I

Lecture 17

Fall 2019 Semester

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Spin and Angular Momentum

- Is orbital angular momentum conserved by Fermions? Does $\vec{L} = \vec{r} \times \vec{p}$ commute with $\widehat{H}$?
- Dirac Hamiltonian: $\widehat{H} = \vec{\alpha} \cdot \vec{p} + \beta m$

$$\left[\widehat{H}, L_x\right] = [\vec{\alpha} \cdot \vec{p} + \beta m, yp_z - zp_y]$$

$$\left[p_y, y\right] = -i$$

$$[p_z, z] = -i$$

$$\left[\widehat{H}, L_x\right] = -i\left(\alpha_y p_z - \alpha_z p_y\right)$$

$$\left[\widehat{H}, \vec{L}\right] = -i(\vec{\alpha} \times \vec{p}) \neq 0$$

- So orbital angular momentum is not a good quantum number.

Spin and Angular Momentum

- Consider the spin operator:

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

- Commutator with the Hamiltonian:

$$[\widehat{H}, \vec{S}] = \frac{1}{2}[\vec{\alpha} \cdot \vec{p} + \beta m, \vec{\Sigma}]$$

$$[\vec{\alpha} \cdot \vec{p}, \vec{\Sigma}] = \left[\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}, \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\right] = 2i(\vec{\alpha} \times \vec{p})$$

$$[\beta, \vec{\Sigma}] = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}\right] = 0$$

$$\left[\widehat{H}, \vec{S}\right] = i(\vec{\alpha} \times \vec{p}) \neq 0$$

- Eigenstates of the spin operator are also not good quantum numbers.

Spin and Angular Momentum

$$\left[\hat{H}, \vec{L}\right] = -i(\vec{\alpha} \times \vec{p}) \neq 0$$

$$\left[\hat{H}, \vec{S}\right] = i(\vec{\alpha} \times \vec{p}) \neq 0$$

- Total angular momentum: $\vec{J} = \vec{L} + \vec{S}$

$$\left[\hat{H}, \vec{J}\right] = 0$$

- Total angular momentum *is* conserved!
- In general, solutions to Dirac's equation are not eigenstates of S_z unless $|\vec{p}| = |p_z|$.
- In this case, $S_z u^{(1)} = \frac{1}{2} u^{(1)},\ S_z u^{(2)} = -\frac{1}{2} u^{(2)}$
- For anti-particles, $S_z v^{(1)} = -\frac{1}{2} v^{(1)},\ S_z v^{(2)} = \frac{1}{2} v^{(2)}$

Helicity

- Solutions to Dirac's equation are eigenstates of helicity:

$$\lambda = \hat{S} \cdot \hat{p} = \frac{\vec{S} \cdot \vec{p}}{|\vec{S}||\vec{p}|}$$

$$\left[\widehat{H}, \vec{S} \cdot \vec{p}\right] = 0$$

- How can we show this?

β and $\vec{S}$ are both diagonal so $\left[\beta, \vec{S} \cdot \vec{p}\right] = 0$

$$\left[\vec{\alpha} \cdot \vec{p}, \vec{S} \cdot \vec{p}\right] = p_i p_j \left[\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix} \right]$$

$$= 2i p_i p_j \varepsilon_{ijk} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = 0$$

- It makes sense to use helicity as a quantum number to describe fermions.

Helicity

- When $\hat{S} \cdot \hat{p}|\psi\rangle = +|\psi\rangle$ we say that the state is "right-handed"
- When $\hat{S} \cdot \hat{p}|\psi\rangle = -|\psi\rangle$ we say that the state is "left-handed"
- We can also assign helicity to particles with spin 1
- Check the definitions carefully, but what we refer to as "right-circular polarized light" has negative helicity while "left-circular polarized light" has positive helicity.
- The electric field is transverse to the momentum, so we call these helicity states of a spin 1 particle "transverse" polarization.
- A massive spin 1 particle can also have helicity 0. We call this "longitudinal" polarization.
- It's all very confusing so be careful…

Helicity Eigenstates

- Suppose that ψ is a helicity eigenstate.
- Then,

$$\hat{S} \cdot \hat{p}\psi = \pm\psi$$

- Solutions to Dirac's equation were written:

$$\psi = N \begin{pmatrix} \chi^{(s)} \\ \dfrac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^{(s)} \end{pmatrix}$$

- In the relativistic limit $|\vec{p}| \rightarrow E$,

$$\psi = N \begin{pmatrix} \chi^{(s)} \\ \vec{\sigma} \cdot \hat{p}\, \chi^{(s)} \end{pmatrix}$$

$$\hat{S} \cdot \hat{p}\psi = N \begin{pmatrix} \vec{\sigma} \cdot \hat{p}\, \chi^{(s)} \\ \chi^{(s)} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \psi$$

Chirality

- Next, we introduce another gamma matrix:

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(in the Dirac representation)

$$(\gamma^5)^2 = 1$$
$$\{\gamma^\mu, \gamma^5\} = 0$$

- Now consider the operators

$$P_L = \frac{1}{2}(1 - \gamma^5)$$
$$P_R = \frac{1}{2}(1 + \gamma^5)$$

- These are projection operators since

$$P_L^2 = P_L, P_R^2 = P_R, P_L P_R = 0 \text{ and } P_L + P_R = 1$$

- How do these projection operators relate to helicity?

Chirality

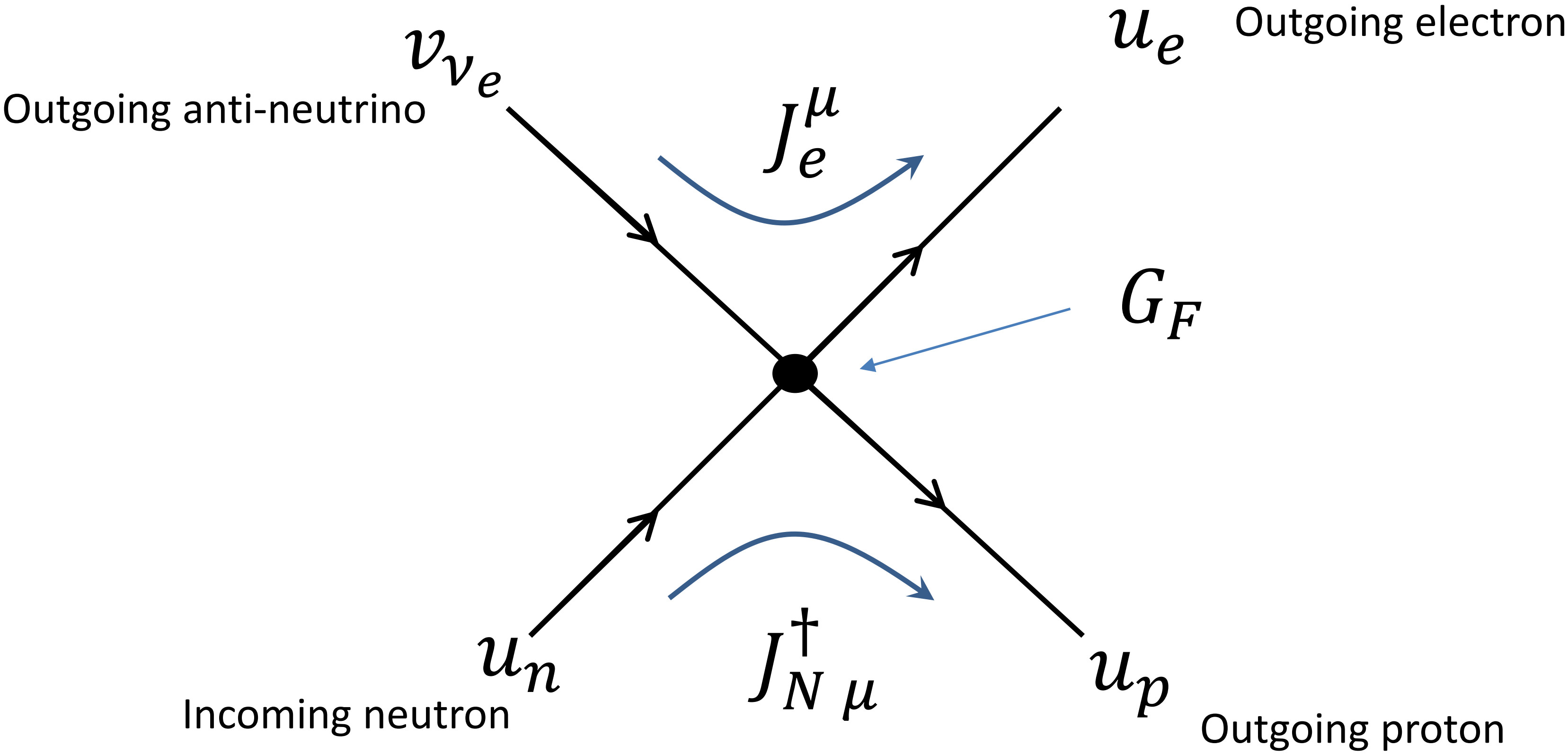
- In the relativistic limit, $P_{R/L}$ project out components of right- and left-handed helicity.
- When this limit is not satisfied, $P_{R/L}$ project out components of right- and left-handed "chirality" but this is not identical to the helicity.
- At high energies, there can be a *very* small right-handed helicity component in a left-handed chiral state.
- This is why, for example, $\pi^+ \to \mu^+\nu_\mu$ is not forbidden and why $\pi^+ \to e^+\nu_\mu$ is very highly suppressed.
- The μ^+ is a right-handed chiral state, but must have left-handed helicity to conserve angular momentum.

Weak Interactions

- Quantum Electrodynamics works extremely well for describing electrons and photons.
- How can we describe weak interactions using a similar formalism, but with no understanding of the dynamics?
- Incoming and outgoing particles are fermions but that's about all we know.
- Fermi parameterized this ignorance by introducing a universal fudge-factor, to characterize the strength of weak interactions.

Weak Interactions

- Consider how we might describe nuclear β-decay
- Fermi 4-point interaction:



Weak Interactions

- The weak interaction couples two “weak currents”:
  - Electron-flavored current

$$J_e^\mu = \overline{u}_e \gamma^\mu v_{\nu_e}$$

  - Nucleon-flavored current

$$J_N^\mu = \overline{u}_p \gamma^\mu u_n$$

- Unlike currents in QED, these change the electric charge of the lepton or nucleon.
- We call these “charged current” weak interactions

Weak Interactions

- This is what Fermi originally proposed, but it was subsequently learned that weak interactions only couple to left-handed fermions.
  - Left-handed in the *CHIRAL* sense, not in the helicity sense
- We can make left-handed chiral currents by inserting the chiral projection operators:

$$J_e^\mu = \overline{u}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\, v_{\nu_e}$$

$$J_N^\mu = \overline{u}_p \gamma^\mu \frac{1}{2}(1-\gamma^5)\, u_n$$

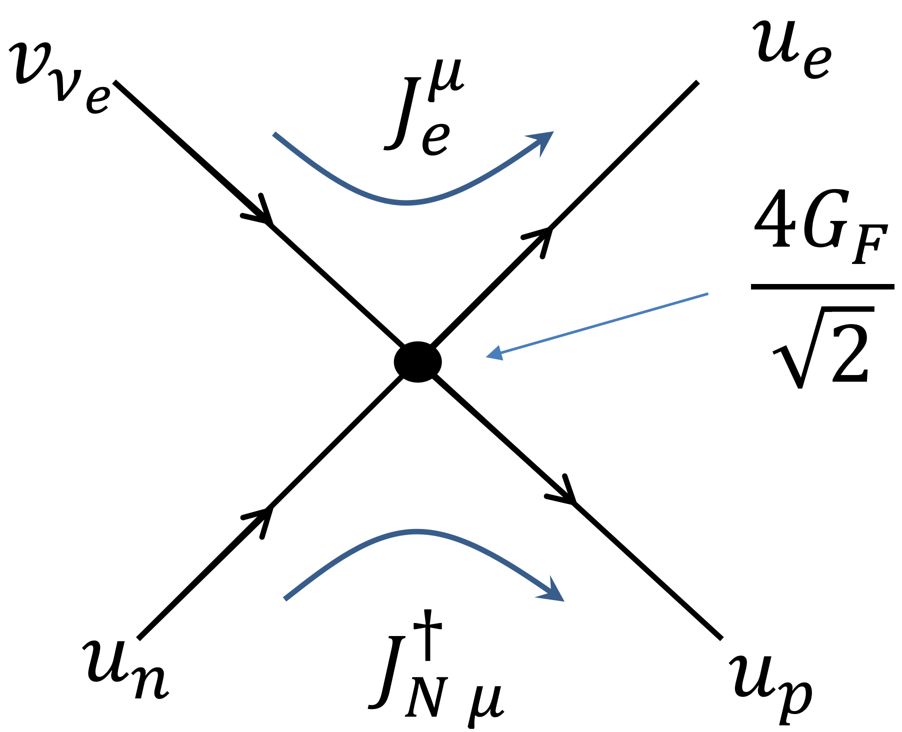
$$\left(J_N^\mu\right)^\dagger = u_n^\dagger \frac{1}{2}\left(1-(\gamma^5)^\dagger\right)(\gamma^\mu)^\dagger \gamma^0 u_p$$

$$= u_n^\dagger \frac{1}{2}(1-\gamma^0\gamma^5\gamma^0)(\gamma^0\gamma^\mu\gamma^0)\gamma^0 u_p$$

$$\overline{u}_n\, \gamma^\mu \frac{1}{2}(1-\gamma^5) u_p$$

Weak Interactions

- With left-handed charged currents, the rules for the 4-point interaction are:



The fudge factor universal coupling had to be redefined to be compatible with historical definitions.

$$J_e^\mu = \overline{u}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\, v_{\nu_e}$$

$$\left(J_N^\mu\right)^\dagger = \overline{u}_n\, \gamma^\mu \frac{1}{2}(1-\gamma^5) u_p$$

Muon Lifetime

- Let's calculate the muon lifetime…
- Alternatively, we can use the muon lifetime as a way to measure G_F

$$\mu^- \to e^- \overline{\nu}_e \nu_\mu$$

  - Electron current:

$$J^\mu_{electron} = \overline{u}_e \gamma^\mu \frac{1}{2}(1-\gamma^5)\, v_{\nu_e}$$

  - Muon current:

$$J^\mu_{muon} = \overline{u}_\mu \gamma^\mu \frac{1}{2}(1-\gamma^5)\, u_{\nu_e}$$

  - Invariant amplitude:

$$-i\mathcal{M} = \frac{4G_F}{\sqrt{2}}\left(J^\mu_{electron}\right)\left(J_{\mu\, muon}\right)^\dagger$$

Muon Lifetime

- Average over initial spins, sum over final spins
  – The chiral projection operator will select *ONLY* the left-handed components

$$\left|\overline{\mathcal{M}}\right|^2 = \frac{G_F}{2} \cdot \frac{1}{2} \sum_{spins} (L_{electron})^{\mu\nu} (L_{muon})_{\mu\nu}$$

$$(L_{electron})^{\mu\nu} = \mathrm{Tr}[u_e \overline{u}_e \gamma^\mu (1-\gamma^5) v_{\nu_e} \overline{v}_{\nu_e} (1+\gamma^5) \gamma^\nu]$$
$$= 2Tr\left[\not{k}_e \gamma^\mu \not{k}_{\nu_e} \gamma^\nu (1-\gamma^5)\right]$$

- Then we can use some more trace identities:

$$Tr[\not{a}\not{b}\not{c}\not{d}\gamma^5] = 4i\varepsilon_{\mu\nu\rho\sigma} a^\mu b^\nu c^\rho d^\sigma$$

Where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely anti-symmetric tensor.

Muon Lifetime

- These calculations can be tedious, but there are programs that can help with evaluating them.
  - For example, Mathematica
- Eventually, it turns out that

$$\left|\overline{\mathcal{M}}\right|^2 = G_F \cdot 64\,(p_\mu \cdot k_{\nu_e})(k_e \cdot k_{\nu_\mu})$$

- Phase space:

$$d\Gamma = \frac{1}{2M}\left|\overline{\mathcal{M}}\right|^2 \frac{dk_{\nu_e}}{(2\pi)^3}\frac{1}{2E_{\nu_e}} \cdot \frac{dk_e}{(2\pi)^3}\frac{1}{2E_e} \cdot \frac{dk_{\nu_\mu}}{(2\pi)^3}\frac{1}{2E_{\nu_\mu}}$$

- Do the integral…

$$\Gamma = \frac{1}{\tau} = \frac{G_F^2 M^5}{192\pi^3}$$

Muon Lifetime

- Using the measured muon properties:

$$\tau_\mu = 2.20 \times 10^{-6}\ \mathrm{s}$$

$$(c\tau_\mu = 659\ \mathrm{m})$$

$$M_\mu = 106\ \mathrm{MeV}/c^2$$

$$\hbar c = 197\ \mathrm{MeV} \cdot \mathrm{fm}$$

$$G_F = \sqrt{\frac{192\pi^3 \cdot (197\ \mathrm{MeV} \cdot \mathrm{fm})}{(659 \times 10^{15}\ \mathrm{fm})(106\,\mathrm{MeV}/c^2)^5}}$$

$$= 1.15 \times 10^{-5}\ \mathrm{GeV}^{-2}$$

Problems with the 4-point Interaction

- Assuming that G_F is a universal coupling, we can also calculate the cross section for neutrino-neutrino scattering

$$\nu_\mu \overline{\nu}_\mu \to \mu^+ \mu^-$$

- We can't actually do this experiment, but we can calculate the cross section.
- Total cross section:

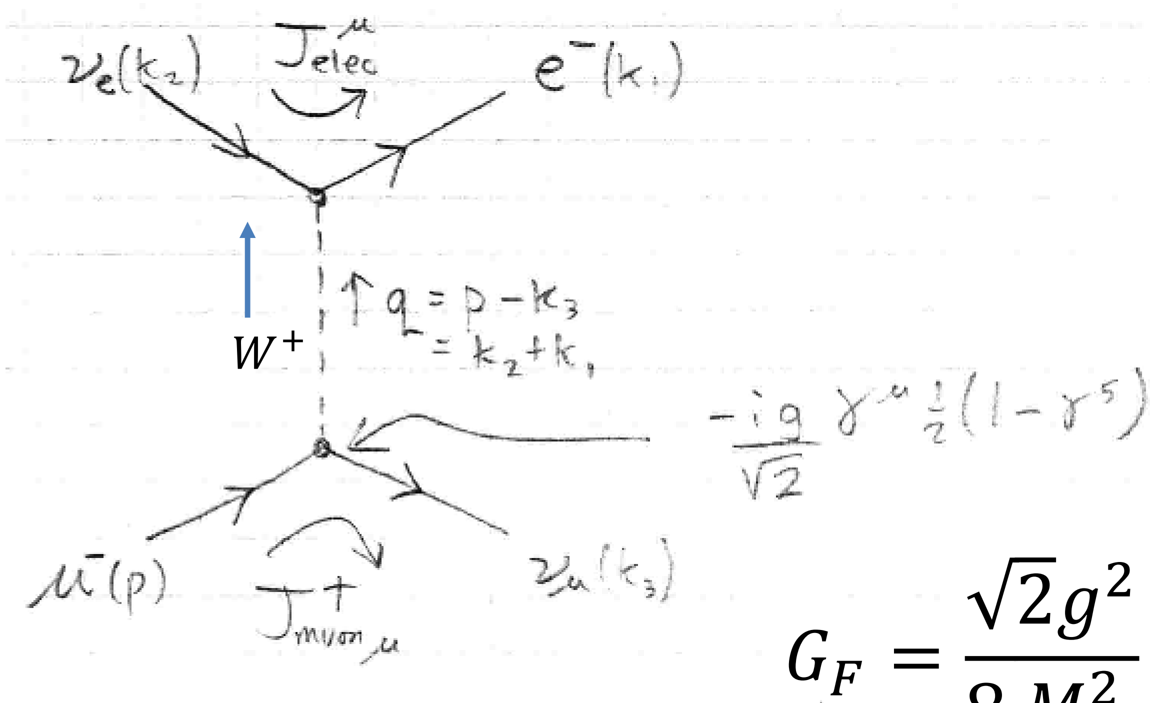
$$\sigma(s) = \frac{G_F^2 s}{6\pi}$$

- The cross section increases quadratically with $E_{cm} = \sqrt{s}$
- At an energy around 100 GeV, this violates unitarity (conservation of probability).

Problems with the 4-point Interaction

- The Fermi 4-point interaction is an example of an *Effective Field Theory*
- It is a low-energy parametrization of a more complete high-energy theory
- Some new physics (new particles) must be introduced at about the mass scale that the low-energy theory stops making sense.
- This is fixed by introducing the W boson…

W-boson couplings



$$G_F = \frac{\sqrt{2} g^2}{8\, M_W^2}$$

The weak interaction is weak, not because the coupling is small ($g \approx e$) but because M_W is large, $\mathcal{O}(100 \text{ GeV})$.