PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 56400
Introduction to Elementary Particle Physics I

Lecture 16

Fall 2019 Semester

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Review of Lecture 15

- When we introduced a (classical) electromagnetic field, the Dirac equation became inhomogeneous:

$$\left(\gamma^\mu p_\mu - m\right)\psi(p) = -e\gamma^\mu A_\mu \psi(p) = \gamma^0 V \psi(p)$$

- We wanted to solve this using time dependent perturbation theory.
- We expressed matrix elements in the basis of asymptotically free states, $|i\rangle$.

$$d\Gamma = \frac{2\pi}{M} \prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} |\langle f|V|i\rangle|^2$$

$$d\sigma = \frac{2\pi}{|v_{AB}| \cdot 2E_A \cdot 2E_B} \prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} |\langle f|V|i\rangle|^2$$

Free Photon States

- Next, we need a way to describe free photons.
- Photons are quanta of the electromagnetic field and are manifest as oscillating $\vec{E}$ and $\vec{B}$ vector field components.
- These components must transform as expected under rotations and Lorentz boosts.
- Photons are massless so there are only two polarization states, right-circular (+) and left-circular (-).
- If we describe a photon state by a 4-vector $\varepsilon^{\mu}(k)$ then we also need to specify constraints to reduce the number of degrees of freedom from 4 to 2.

Free Photon States

- The $\vec{E}$ and $\vec{B}$ fields can be expressed in terms of the vector potential:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

- Maxwell's equations:

$$\partial_\mu F^{\mu\nu} = j^\nu$$

- Gauge condition:

$$\partial_\mu A^\mu = 0$$

- In free space, $j^\nu = 0$

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu$$

Free Photon States

- We can look for solutions that are of the form

$$A^{\mu} = \varepsilon^{\mu} e^{-ik\cdot x}$$

- Gauge condition:

$$\partial_{\mu} A^{\mu} = k_{\mu} \varepsilon^{\mu} e^{-ik\cdot x} = 0$$

- Polarization vectors:

$$\varepsilon^{(+)} = -(0,1,i,0)/\sqrt{2}$$
$$\varepsilon^{(-)} = (0,1,-i,0)/\sqrt{2}$$

- Inhomogeneous equation:

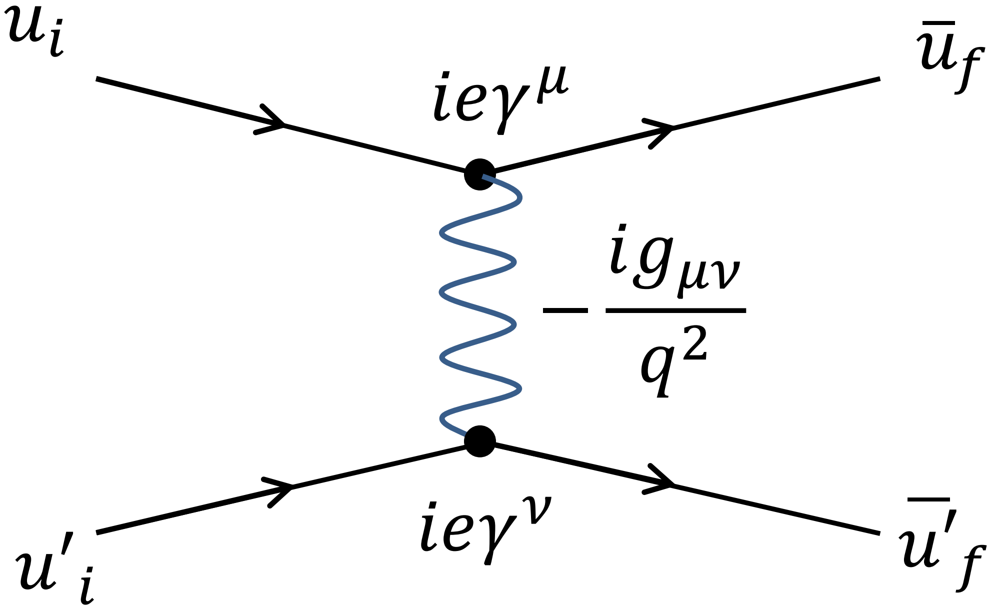
$$\partial_{\mu} F^{\mu\nu} = \partial_{\mu} \partial^{\mu} A^{\nu} = j^{\nu} = -e(\bar{\psi}_f \gamma^{\nu} \psi_i)$$

- Fourier transform:

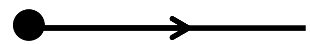
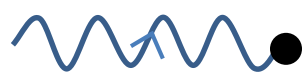
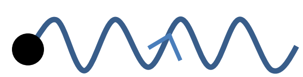
$$q^2 A^{\nu} = -e\left(\overline{u}_f \gamma^{\nu} u_i\right)$$

Feynman Rules for QED

- An electron current creates an electromagnetic field, which interacts with another electron current:

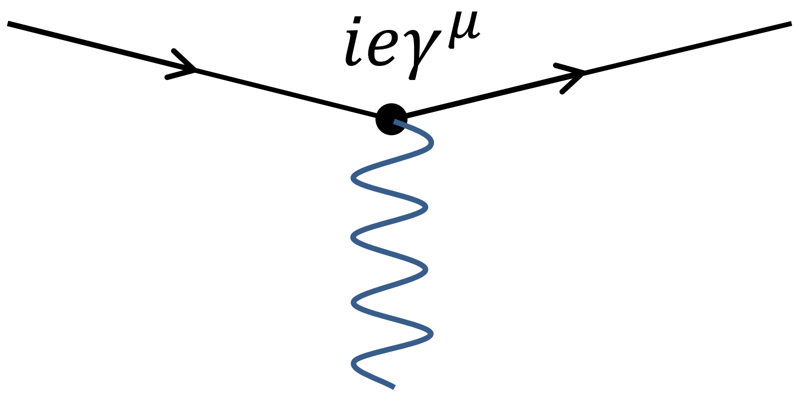


Feynman Rules for QED

- The structure of the transition amplitude can be expressed graphically using the following rules:
  - External fermions:
    - Incoming fermion u 
    - Outgoing fermion  $\bar{u}$
  - External anti-fermions:
    - Incoming anti-fermion $\bar{v}$ 
    - Outgoing anti-fermion  v
  - External photons
    - Incoming photon ε_μ 
    - Outgoing photon  ε_μ^*

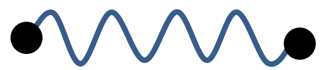
Feynman Rules for QED

- The structure of the transition amplitude can be expressed graphically using the following rules:
  - There is only one kind of vertex in QED:



Feynman Rules for QED

- The structure of the transition amplitude can be expressed graphically using the following rules:
  - Internal photon line (photon propagator):



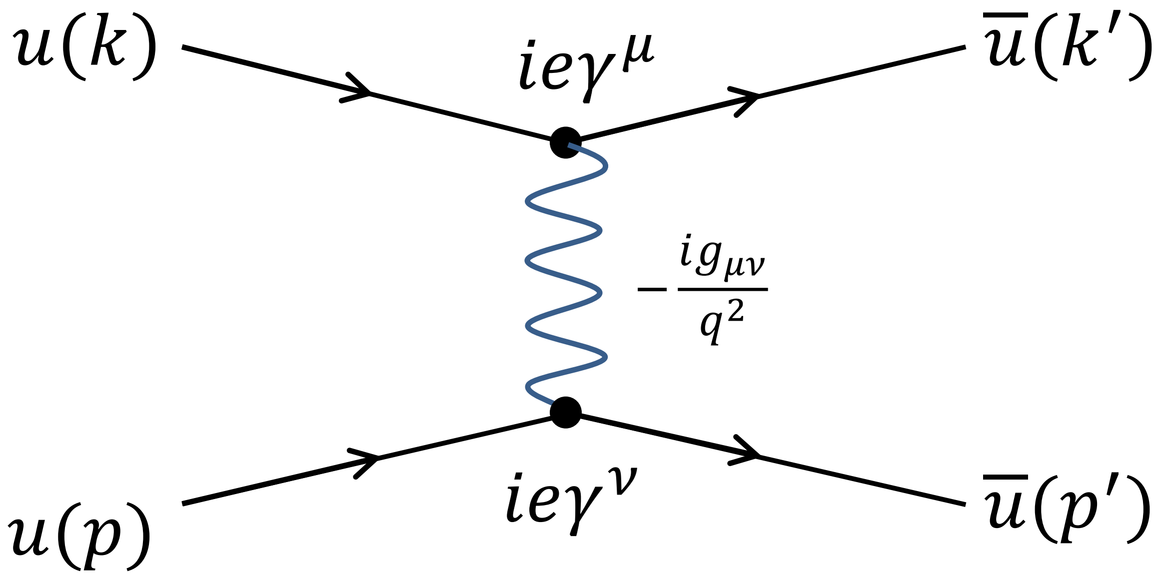
$$-\frac{ig_{\mu\nu}}{q^2}$$

  - Internal fermion line (fermion propagator):

$$\frac{i(\not{p} + m)}{p^2 - m^2}$$

Examples of Feynman Amplitudes

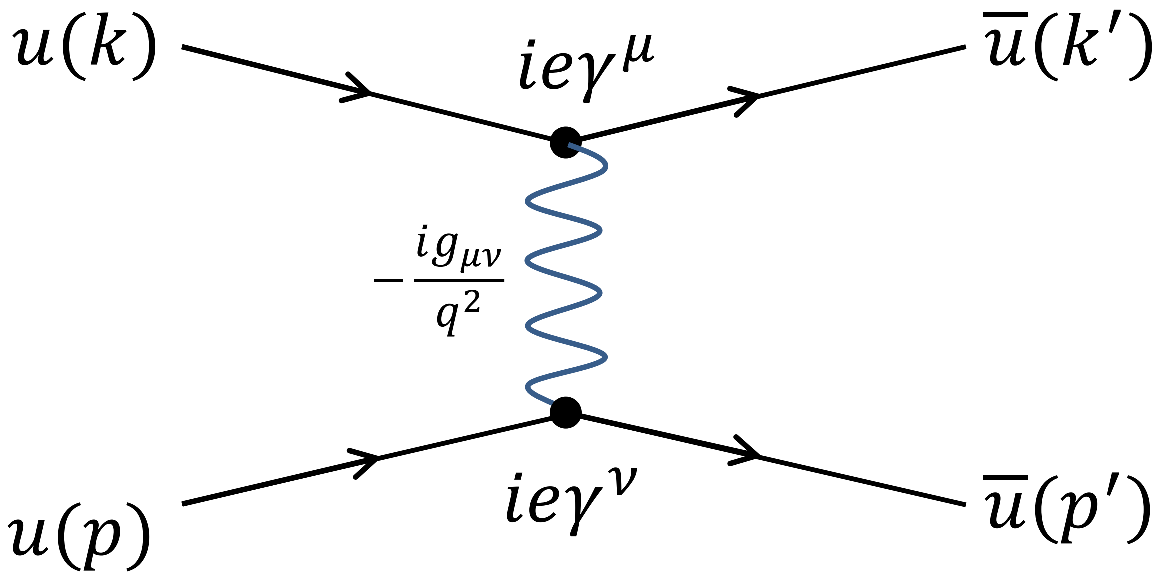
- Electron-muon scattering



$$-i\mathcal{M} = \overline{u}(k')(ie\gamma^{\mu})u(k)\left(-\frac{ig_{\mu\nu}}{q^2}\right)\overline{u}(p')(ie\gamma^{\nu})u(p)$$

Examples of Feynman Amplitudes

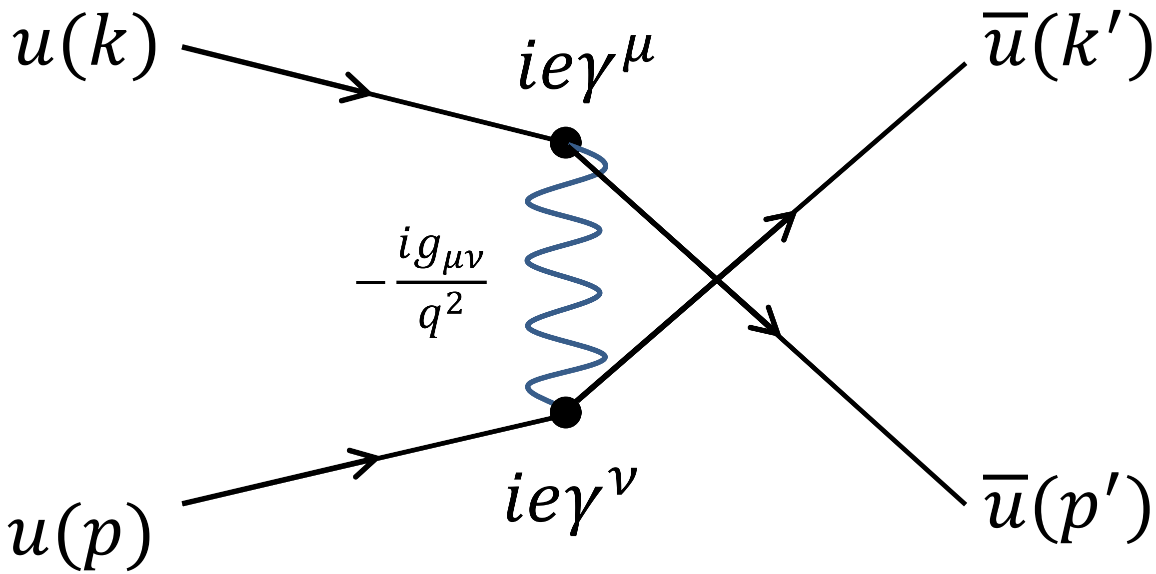
- Electron-electron scattering



$$-i\mathcal{M}_1 = \overline{u}(k')(ie\gamma^{\mu})u(k)\left(-\frac{ig_{\mu\nu}}{q^2}\right)\overline{u}(p')(ie\gamma^{\nu})u(p)$$

Examples of Feynman Amplitudes

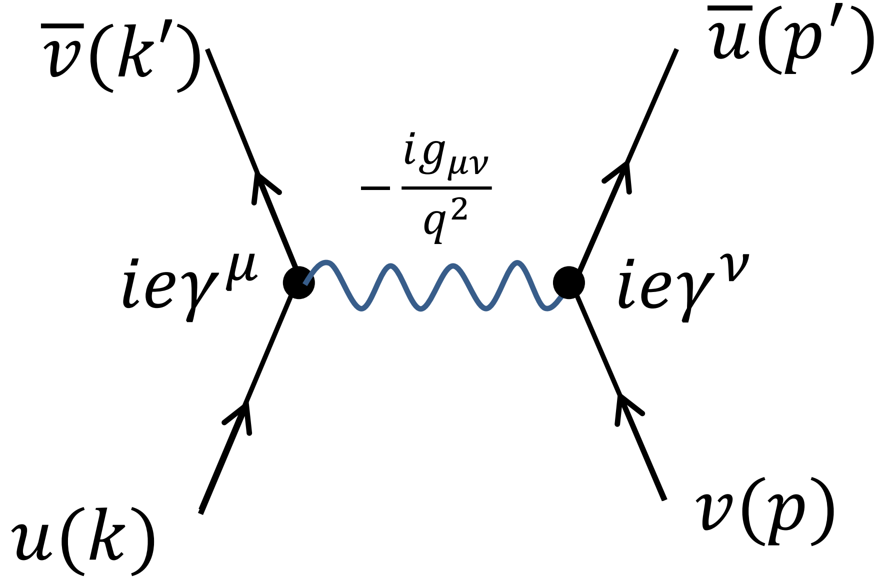
- Electron-electron scattering



$$-i\mathcal{M}_2 = \overline{u}(p')(ie\gamma^{\mu})u(k)\left(-\frac{ig_{\mu\nu}}{q^2}\right)\overline{u}(p')(ie\gamma^{\nu})u(p)$$

Examples of Feynman Amplitudes

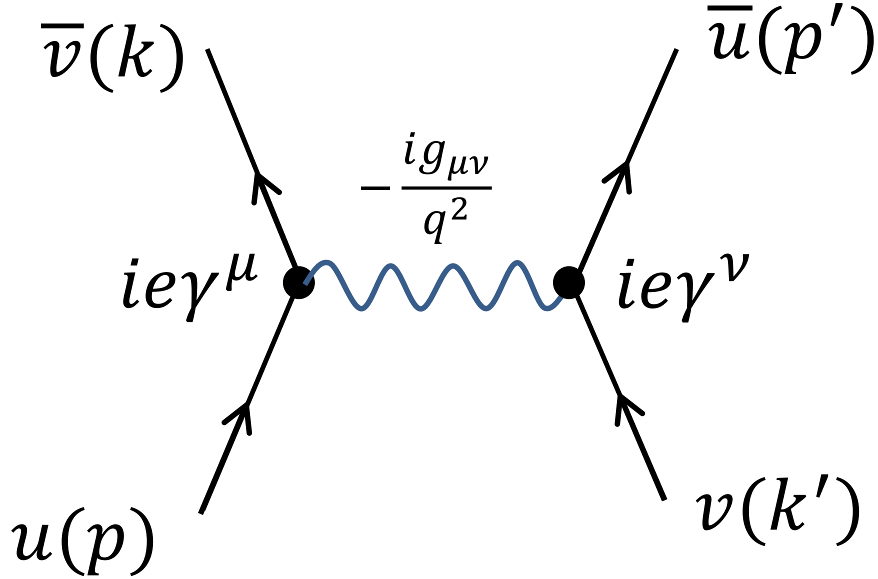
- $e^+e^- \to f\overline{f}$ (but not $e^+e^- \to e^+e^-$)



$$-i\mathcal{M} = \overline{v}(k)(ie\gamma^\mu)u(k)\left(-\frac{ig_{\mu\nu}}{q^2}\right)\overline{u}(p')(ie\gamma^\nu)v(p)$$

Examples of Feynman Amplitudes

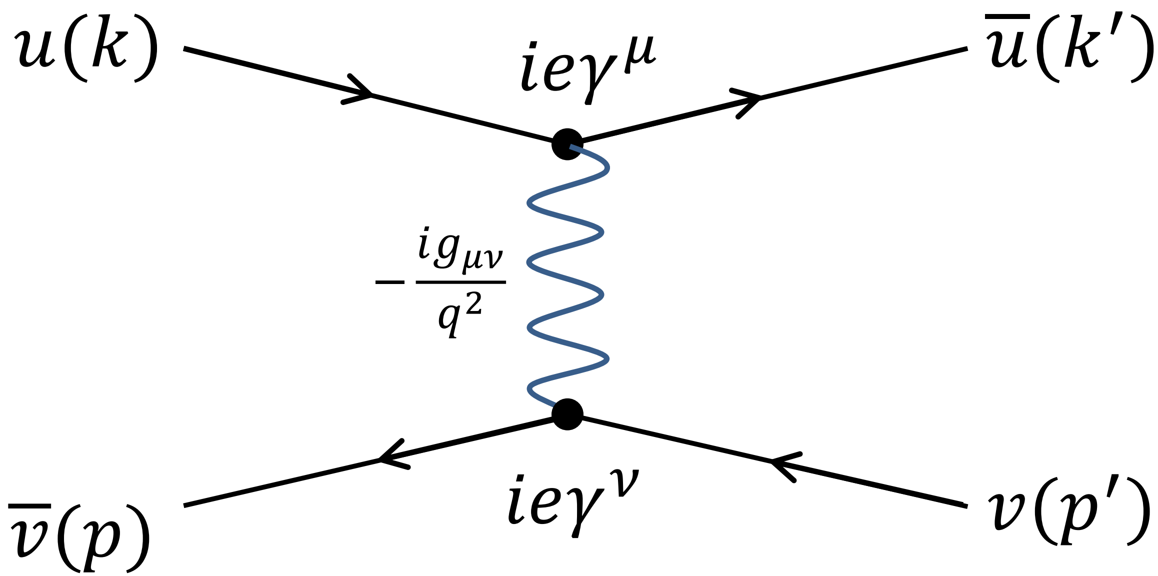
- Bhabha scattering: $e^+e^- \to e^+e^-$



$$-i\mathcal{M}_1 = \overline{v}(k)(ie\gamma^\mu)u(p)\left(-\frac{ig_{\mu\nu}}{q^2}\right)\overline{u}(p')(ie\gamma^\nu)v(k')$$

Examples of Feynman Amplitudes

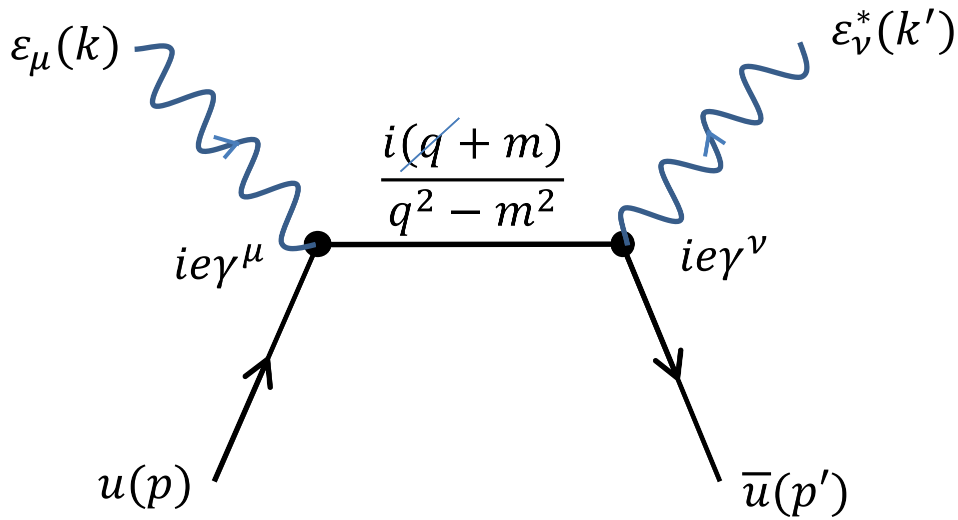
- Bhabha scattering: $e^+e^- \to e^+e^-$



$$-i\mathcal{M}_2 = \overline{u}(k')(ie\gamma^\mu)u(k)\left(-\frac{ig_{\mu\nu}}{q^2}\right)\overline{v}(p)(ie\gamma^\nu)v(p')$$

Examples of Feynman Amplitudes

- Compton scattering:

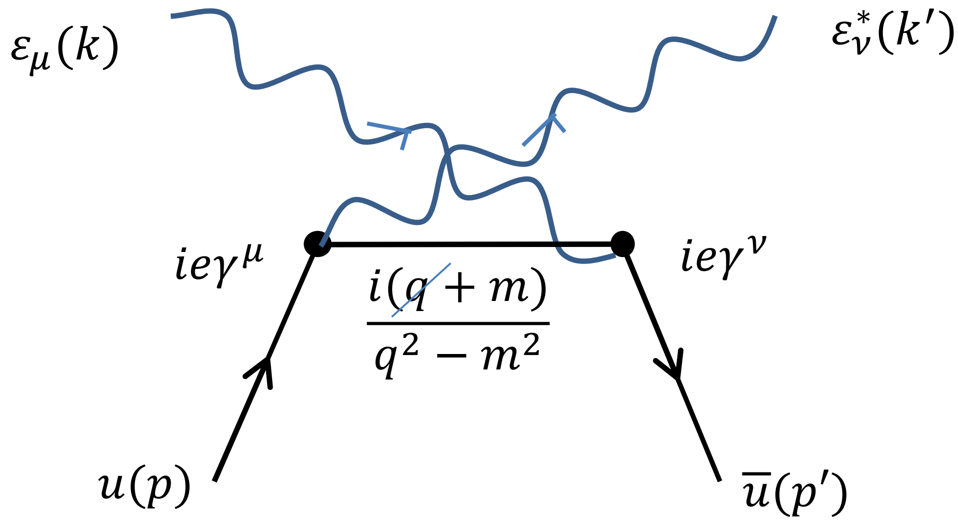


$$-i\mathcal{M}_1 = \overline{u}(p')(ie\gamma^\nu)\left(\frac{i(\not{q}+m)}{q^2-m^2}\right)(ie\gamma^\mu)u(p)\varepsilon_\nu^*(k')\varepsilon_\mu(k)$$

$$q = p + k = p' + k'$$

Examples of Feynman Amplitudes

- Compton scattering:



$$-i\mathcal{M}_2 = \overline{u}(p')(ie\gamma^\nu)\left(\frac{i(\not{q}+m)}{q^2-m^2}\right)(ie\gamma^\mu)u(p)\varepsilon_\nu^*(k')\varepsilon_\mu(k)$$

$$q = p - k' = p' + k$$

Probability Amplitudes

- Expressions for cross sections and decay rates include the term $|\mathcal{M}|^2$
- When more than one Feynman diagram contributes the amplitudes are summed:

$$|\mathcal{M}|^2 = |\mathcal{M}_1 + \mathcal{M}_2|^2 = |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\mathrm{Re}(\mathcal{M}_1^* \mathcal{M}_2)$$

- Sum over initial and final state spins/polarizations:

$$\sum_s u^{(s)}(p)\bar{u}^{(s)}(p) = p^\mu \gamma_\mu + m$$

$$\sum_s v^{(s)}(p)\bar{v}^{(s)}(p) = p^\mu \gamma_\mu - m$$

$$\sum_\lambda \varepsilon_\mu^{(\lambda)^*} \varepsilon_\nu^{(\lambda)} = -g_{\mu\nu}$$

Probability Amplitudes

- The spin-averaged matrix elements can be simplified using trace algebra.
- For electron-muon scattering

$$-i\mathcal{M} = \overline{u}(k')(ie\gamma^{\mu})u(k)\left(-\frac{ig_{\mu\nu}}{q^2}\right)\overline{u}(p')(ie\gamma^{\nu})u(p)$$

$$\left|\overline{\mathcal{M}}\right|^2 = \frac{e^4}{q^4}\sum_{spins}\left(\overline{u}(k')\gamma^{\mu}u(k)\overline{u}(k)\gamma^{\nu}u(k')\right)\cdot\left(\overline{u}(p')\gamma_{\mu}u(p)\overline{u}(p)\gamma_{\nu}u(p')\right)$$

$$= \frac{e^4}{q^4}\mathrm{Tr}\left((\not{k}' + m)\gamma^{\mu}(\not{k} + m)\gamma^{\nu}\right)\cdot\mathrm{Tr}\left((\not{p}' + M)\gamma_{\mu}(\not{p} + M)\gamma_{\nu}\right)$$

- This needs to be divided by 4 to average over initial state spins.

Probability Amplitudes

- Anti-commutation relations:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

- Trace theorems:
  - Trace of an odd number of gamma matrices vanishes
  - $Tr(I) = 4$
  - $Tr(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$ → $Tr(\not{a}\not{b}) = 4a \cdot b$
  - $Tr(\not{a}\not{b}\not{c}\not{d}) = 4[(a \cdot b)(c \cdot d) - (a \cdot c)(b \cdot d) + (a \cdot d)(b \cdot c)]$
  - $Tr(\not{a}\gamma^\mu\not{b}\gamma^\nu) = 4[a^\mu b^\nu + a^\nu b^\mu - a \cdot b g^{\mu\nu}]$

Probability Amplitudes

- At high energies we might ignore the masses.
- Electron-muon scattering:

$$\left|\overline{\mathcal{M}}\right|^2 = \frac{8e^4}{q^4}\left[(k' \cdot p')(k \cdot p) + (k' \cdot p)(k \cdot p')\right]$$

- Mandlestam variables:

$$s \equiv (k+p)^2 \approx 2k \cdot p \approx 2k' \cdot p'$$
$$t \equiv (k-k')^2 \approx -2k \cdot k' \approx -2p \cdot p'$$
$$u \equiv (k-p')^2 \approx -2k \cdot p' \approx -2k' \cdot p$$

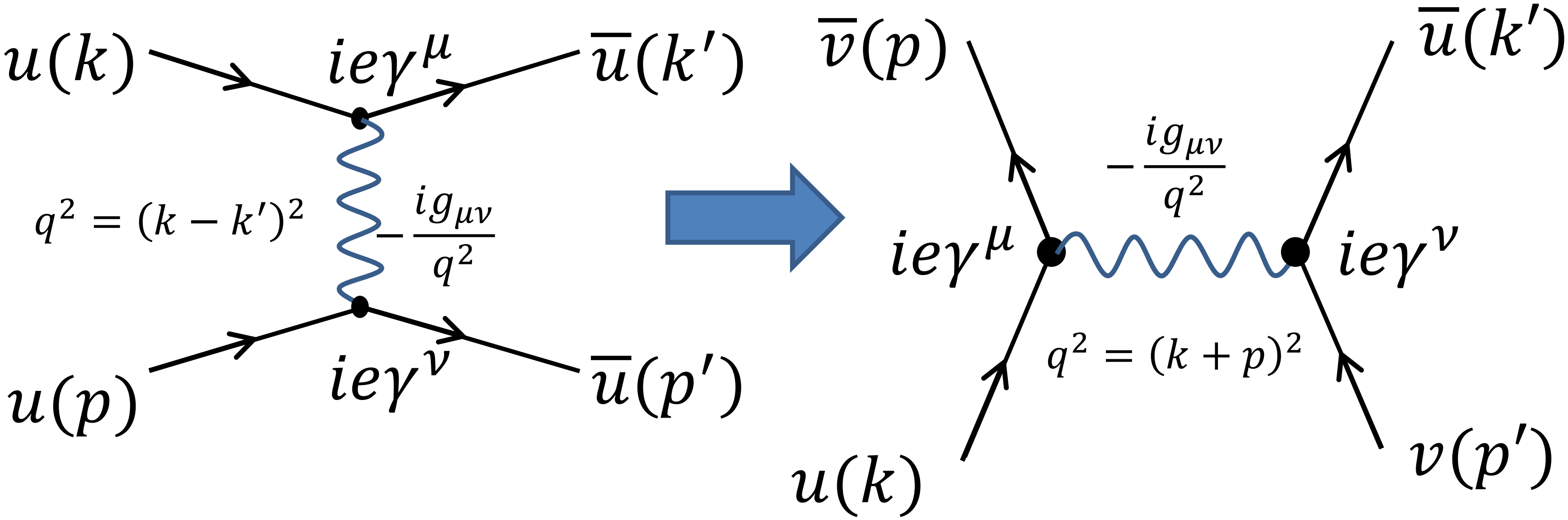
- Recall that in this case, $q^2 = (k-k')^2 = t$

Probability Amplitudes

- Electron-muon scattering at high energies:

$$\left|\overline{\mathcal{M}}\right|^2 = 2e^4 \frac{s^2 + u^2}{t^2}$$

- Other probability amplitudes can be deduced by relabeling the 4-vectors:



Probability Amplitudes

- Fermion pair-production by e^+e^- scattering:

$$\left|\overline{\mathcal{M}}\right|^2 = 2e^4 \frac{t^2 + u^2}{s^2}$$

- Center-of-mass energy is $E_{cm} = \sqrt{s}$
- Assume the initial particles collide along the z-axis

$$t = -2k \cdot k' = -2s(1 - \cos\theta)$$

$$u = -2k \cdot p' = -2s(1 + \cos\theta)$$

$$\left|\overline{\mathcal{M}}\right|^2 = 16e^4(1 + \cos^2\theta)$$

Probability Amplitudes

- Fermion pair-production by e^+e^- scattering:

$$\left|\overline{\mathcal{M}}\right|^2 = 2e^4 \frac{t^2 + u^2}{s^2}$$

- Phase-space for 2→2 scattering in the center-of-mass frame:

$$dQ = \frac{1}{4\pi^2} \frac{p_f}{4\sqrt{s}} d\Omega$$

- Initial flux in the center-of-mass frame:

$$F = 4p_i\sqrt{s}$$

- Differential cross section:

$$d\sigma = \frac{\left|\overline{\mathcal{M}}\right|^2}{F} dQ = \frac{16e^4}{64\pi^2 s} \frac{p_f}{p_i} \cdot \left(1 + \cos^2\theta\right) d\Omega$$

Differential Cross Sections

- Fermion anti-fermion pair production:

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1 + \cos^2\theta)$$

- Total cross section:

$$\sigma(e^+e^- \to f\overline{f}) = \frac{4\pi\alpha^2}{3s}$$

- This applies when $f = \mu, \tau$ but when the final state consists of quarks, we need to fix this up.

Hadronic Cross Section

- Quarks can have three colors, so we have to sum over these final states.
- Quarks have fractional charges which modifies *one* of the vertex factors.

$$\sigma(e^+e^- \to q\overline{q}) = 3Q_q^2 \times \frac{4\pi\alpha^2}{3s}$$

- Quarks will eventually turn into hadrons, so we just need to count events containing hadrons.
- Unfortunately, we can't tell what kind of quark was produced (unless we are clever) so we need to sum over all possible quark flavors.

Hadronic Cross Section

- Consider the ratio of cross sections:

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_q Q_q^2$$

- If the center-of-mass energy is insufficient to produce charm quarks, then

$$R = 3\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right) = 2$$

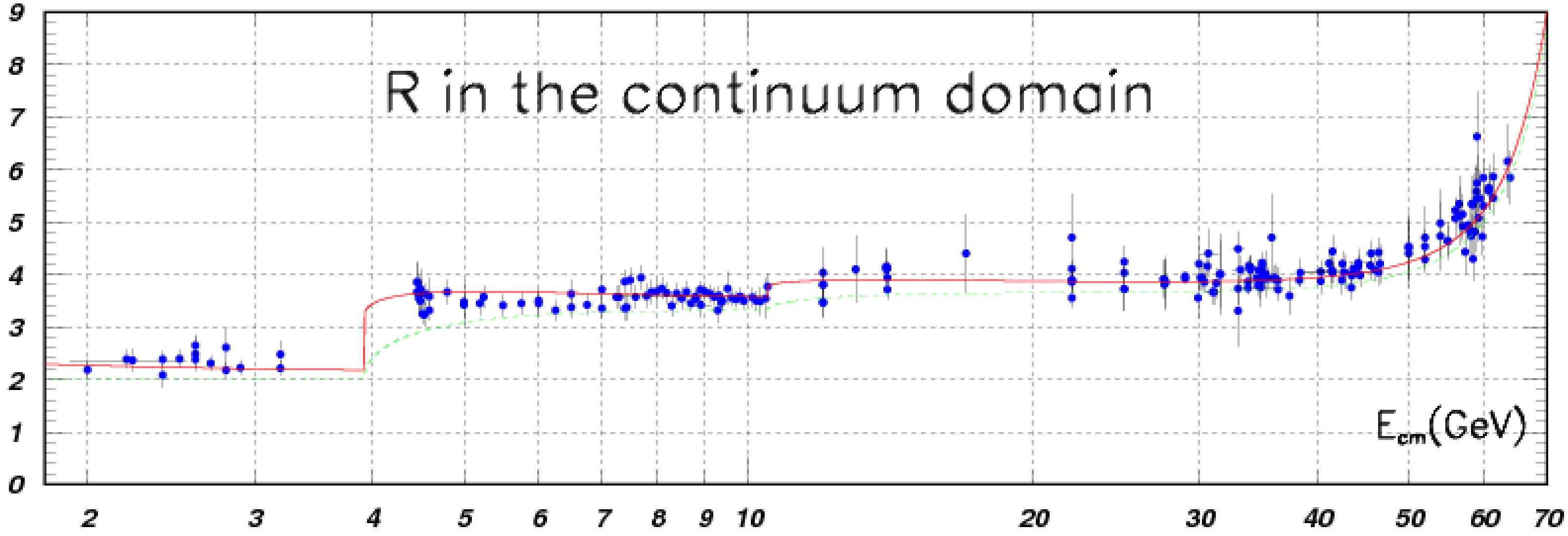
- After crossing the charm production threshold,

$$R = 3\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2\right) = \frac{10}{3}$$

- After crossing the $b\bar{b}$ production threshold,

$$R = 3\left(\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2\right) = \frac{11}{3}$$

$e^+e^- \to$ hadrons



- The simple picture is modified slightly due to
  - Threshold effects
  - Resonances
  - Radiative corrections (which can be calculated)
- Deviations from this picture must be due to new physics not included in the model.