Purdue University Department of Physics

Physics 56400

Introduction to Elementary Particle Physics I

Lecture 15

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Interaction with an External Field

- Schrodinger's equation for a free particle:

$$\hat{H}\Psi = \frac{\hat{p}^2}{2m}\Psi$$

- Schrodinger's equation for a particle subjected to a force:

$$\hat{H}\Psi = \left(\frac{\hat{p}^2}{2m} + V\right)\Psi$$

$$\vec{F} = -\nabla V$$

- Hydrogen atom:

$$V(r) = -\frac{Ze^2}{r}$$

$$F_r = -\frac{\partial V}{\partial r} = -\frac{Ze^2}{r^2}$$

$$\hat{H}\Psi = \left(\frac{\hat{p}^2}{2m} - \frac{Ze^2}{r}\right)\Psi$$

Interaction with an External Field

- Minimal substitution:

$$p^\mu \to p^\mu - qA^\mu$$
$$= p^\mu + eA^\mu \text{ (for an electron with } q = -e\text{)}$$

- Vector potential,

$$A^\mu = \left(\phi, \vec{A}\right)$$

- Hydrogen atom:

$$\phi = \frac{Ze}{r}$$

- Schrodinger's equation:

$$\hat{H}\Psi = \left(\frac{\hat{p}^2}{2m} + e\phi\right)\Psi = \left(\frac{\hat{p}^2}{2m} - \frac{Ze^2}{r}\right)\Psi$$

Interaction with an External Field

- Dirac equation (homogeneous):

$$\left(\gamma^{\mu} p_{\mu} - m\right)\psi(p) = 0$$

- Minimal substitution:

$$p^{\mu} \rightarrow p^{\mu} + eA^{\mu}$$

- Dirac equation with an external field:

$$\left(\gamma^{\mu} p_{\mu} - m\right)\psi(p) = -e\gamma^{\mu} A_{\mu}\psi(p)$$
$$= \gamma^{0} V \psi(p)$$

- We can solve this equation using perturbation theory by expanding in powers of e.

Time Dependent Perturbation Theory

- Suppose that $|i\rangle$ is an eigenstate of a "free" Hamiltonian, $\widehat{H}_0$:

$$\widehat{H}_0|i\rangle = E_i|i\rangle$$

- The time evolution operator gives the state at a later time:

$$|i,t\rangle = U(t)|i\rangle = e^{-i\widehat{H}_0 t}|i\rangle = e^{-iE_i t}|i\rangle$$

- For an unstable particle with decay rate Γ, the time evolution operator can be written

$$U(t) = e^{-i\widehat{H}_0 t - \Gamma t/2}$$

- The part that is responsible for the decay is written as a perturbation to the free Hamiltonian

$$\widehat{H} = \widehat{H}_0 + \widehat{V}$$

Time Dependent Perturbation Theory

- We want to find an expression for the state at time t given that it was in state $|i\rangle$ at time $t = 0$.
- Write this in terms of free particle states at time t:

$$|f, t\rangle = e^{-iE_f t}|f\rangle$$

$$|i, t\rangle = \sum_n |n, t\rangle\langle n, t|i, t\rangle$$

$$= \sum_n |n, t\rangle\langle n, t|U(t)|i\rangle$$

$$= \sum_n c_n(t)\,|n, t\rangle$$

Time Dependent Perturbation Theory

$$c_n(t) = \langle n, t|U(t)|i\rangle$$

- This is written in terms of free particle states so

$$\langle n, t| = e^{iE_n t}\langle n|$$

- The time evolution operator is written using the full Hamiltonian:

$$U(t) = e^{-i\hat{H}t} = e^{-i(\hat{H}_0+\hat{V})t}$$

- Thus,

$$\frac{dc_n}{dt} = iE_n\langle n, t|U(t)|i\rangle - i\langle n, t|\left(\hat{H}_0 + \hat{V}\right)U(t)|i\rangle$$
$$= -i\langle n, t|\hat{V}U(t)|i\rangle = -i\langle n, t|\hat{V}|i, t\rangle$$

$$= -\sum_m i\langle n, t|\hat{V}|m, t\rangle\langle m, t|i, t\rangle = -\sum_m i\langle n, t|\hat{V}|m, t\rangle c_m(t)$$

Time Dependent Perturbation Theory

- When $t \to 0$ we expect that $c_i(t) \to e^{-iE_i t - \Gamma t/2}$ and $c_j(t) \to 0$ when $i \neq j$

$$\frac{dc_n}{dt} = -ie^{i(E_n - E_i)t} e^{-\Gamma t/2} \langle n | \hat{V} | i \rangle$$

$$c_n(t) = -i \int_0^t e^{i(E_n - E_i)t'} e^{-\frac{\Gamma t'}{2}} \langle n | \hat{V} | i \rangle dt'$$

- But, if $\langle n | \hat{V} | i \rangle = V_{ni}$ does not depend too strongly on energy then we can take it out of the integral:

$$c_n(t) = -iV_{ni} \int_0^t e^{i(E_n - E_i)t'} e^{-\frac{\Gamma t'}{2}}$$

$$= -i \frac{V_{ni}}{i(E_n - E_i) - \Gamma/2} e^{i(E_n - E_i)t} e^{-\frac{\Gamma t}{2}} \Big|_0^t$$

Time Dependent Perturbation Theory

- What we really care about is the asymptotic limit as $t \to \infty$

$$c_n(\infty) = i \frac{V_{ni}}{i(E_n - E_i) - \Gamma/2}$$

- The probability of observing the final state $|f\rangle$ at some time in the distant future is

$$\mathcal{P}(i \to f) = \left|c_f(\infty)\right|^2 = \frac{\left|V_{fi}\right|^2}{\left(E_f - E_i\right)^2 + (\Gamma/2)^2}$$

Time Dependent Perturbation Theory

- In practice, we are usually interested in the probability of observing a particular class of final states.
- We need to sum over all relevant kinematic configurations of final states, f:

$$\mathcal{P}(i \to f) = \sum_f \left|c_f(\infty)\right|^2 = \int_0^\infty \frac{\rho_f(E)\left|V_{fi}\right|^2}{(E - E_i)^2 + (\Gamma/2)^2} dE$$

- $\rho_f(E)$ is the density of final states with energy E.

Time Dependent Perturbation Theory

- The integrand is very small except when $E \approx E_i$ so

$$\mathcal{P}(i \to f) = \rho_f(E)\left|V_{fi}\right|^2 \int_{-\infty}^{\infty} \frac{dE}{(E - E_i)^2 + (\Gamma/2)^2}$$

$$= \rho_f(E)\left|V_{fi}\right|^2 \cdot \frac{2\pi}{\Gamma}$$

- If we sum over all possible final states we expect that

$$\sum_f \mathcal{P}(i \to f) = 1$$

- The individual probabilities are the branching fractions with partial widths

$$\Gamma_f = \Gamma\, Br(i \to f)$$

- The total decay rate is

$$\Gamma = \sum_f \Gamma_f = \sum_f 2\pi\, \rho_f(E)\left|V_{fi}\right|^2$$

Density of States

- How many states are there with an energy between E and $E + dE$?
- Consider a particle confined to a box with sides of length L.

$$\phi(\vec{x}) = \langle \vec{x} | \phi \rangle = N e^{i\vec{k}\cdot\vec{x}}$$
$$\phi(\vec{x}, t) = N e^{i\vec{k}\cdot\vec{x} - iEt}$$

- Periodic boundary conditions:

$$k_x = 2\pi n_x / L$$
$$k_y = 2\pi n_y / L$$
$$k_z = 2\pi n_z / L$$

Density of States

$$\Delta n_x \Delta n_y \Delta n_z = \frac{\Delta k_x \Delta k_y \Delta k_z}{(2\pi)^3} L^3$$

- Probability density is

$$\rho = i\left(\phi^*(x)\frac{\partial\phi}{dt} - \phi(x)\frac{\partial\phi^*}{dt}\right) = 2E|N|^2$$

- We integrate this over the whole box:

$$\int d^3x\, \rho(x) = 2E|N|^2 L^3$$

- A suitable normalization would be

$$N = 1/\sqrt{V}$$

Density of States

- The number of states per volume is $d^3k L^3/(2\pi)^3$
- Particles per volume is $2E$
- Number of final states per particle is $\frac{d^3k}{(2\pi)^3} \cdot \frac{L^3}{2E}$
- In general,

$$\rho_f = \frac{d^3p_1 L^3}{(2\pi)^3 2E_1} \cdot \frac{d^3p_2 L^3}{(2\pi)^3 2E_2} \cdots \ \delta(E_f - E_i)$$

- The factors of L^3 will cancel with the normalization of states in the matrix element $\langle f|V|i\rangle$.

Transition Amplitude

- The energy-conserving delta function and V_{fi} can be combined into the "transition matrix element":

$$T_{fi} = -i\langle f|\hat{V}|i\rangle \int e^{i(E_f - E_i)t} dt$$

- In the position representation:

$$T_{fi} = -i \int d^3x d^3x' \langle f|x'\rangle\langle x'|\hat{V}|x\rangle\langle x|i\rangle \int e^{i(E_f - E_i)t} dt$$

- If, as is often the case, the operator $\hat{V}$ is local, then

$$\langle x'|\hat{V}|x\rangle = \delta(x' - x)\langle x|\hat{V}|x\rangle$$

- The transition matrix element can be written

$$T_{fi} = -i \int d^3x \, \phi_f^*(\vec{x})V(\vec{x})\phi_i(\vec{x}) \int e^{i(E_f - E_i)t} dt = -i \int d^4x \phi_f^*(x)V(x)\,\phi_i(x)$$

Cross Sections and Decay Rates

- Differential decay rate:

$$d\Gamma = \frac{2\pi}{M} \prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} |\langle f|V|i\rangle|^2$$

- Differential cross section:

$$d\sigma = \frac{2\pi}{|v_{AB}| \cdot 2E_A \cdot 2E_B} \prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} |\langle f|V|i\rangle|^2$$

Transition Amplitude for Spin ½ Particles

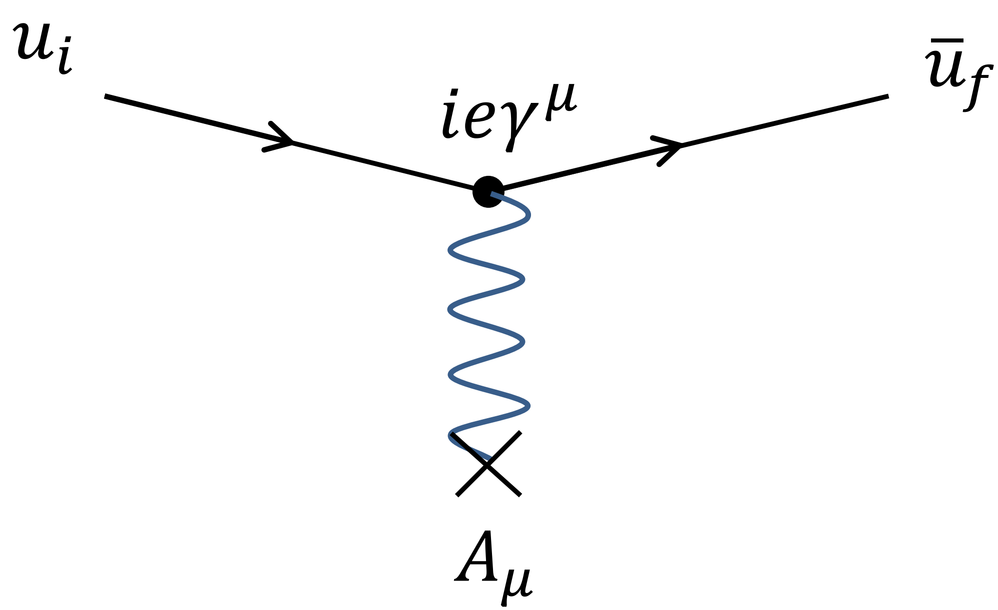
- The free particle solutions to the Dirac equation are
$$\psi(x) = u(p)e^{-ip\cdot x}$$
- The free particle solutions to the adjoint Dirac equation are
$$\bar{\psi}(x) = \bar{u}(p)e^{ip\cdot x}$$
- The perturbation to the Hamiltonian is
$$V(x) = e\gamma^0\gamma^\mu A_\mu(x)$$
- The transition matrix element is

$$T_{fi} = -ie\int d^4x \left(\bar{\psi}_f(x)\gamma^\mu\psi_i(x)\right) A_\mu(x)\, e^{ix\cdot(p_f-p_i)}$$

$$= -ie\bar{u}\left(p_f\right)\gamma^\mu u(p_i)A_\mu(q)$$
$$q = p_f - p_i$$

Transition Amplitudes for Spin ½ Particles

- We can describe the form of this expression graphically:



Coulomb Scattering

- Consider scattering from a static central potential:

$$A^{\mu}(x) = \left(\phi(x), \vec{0}\right)$$

$$\phi(x) = \frac{Ze}{r}$$

- Fourier transform:

$$A^{0}(q) = \frac{Ze}{q^2}$$

- Transition matrix element:

$$T_{fi} = \frac{Ze^2 \bar{u}_f \gamma^0 u_i}{q^2}$$

Coulomb Scattering

- To calculate the probability we need $\left|T_{fi}\right|^2$

$$\left|T_{fi}\right|^2 = \frac{Z^2 e^4}{q^4}\left(\bar{u}_f \gamma^0 u_i\right)^{\dagger}\left(\bar{u}_f \gamma^0 u_i\right)$$

$$= \frac{Z^2 e^4}{q^4}\left(\bar{u}_i \gamma^0 u_f\right)\left(\bar{u}_f \gamma^0 u_i\right)$$

$$= \frac{Z^2 e^4}{q^4}\,\mathrm{Tr}(\gamma^0 u_f \bar{u}_f \gamma^0 u_i \bar{u}_i)$$

Coulomb Scattering

- We can try to evaluate this explicitly:

$$u_i \bar{u}_i = \begin{pmatrix} (E_i + m)\chi^{(s)}{\chi^{(s)}}^T & -\vec{\sigma}\cdot\vec{p}_i\chi^{(s)}{\chi^{(s)}}^T \\ \vec{\sigma}\cdot\vec{p}_i\chi^{(s)}{\chi^{(s)}}^T & -(E_i - m)\chi^{(s)}{\chi^{(s)}}^T \end{pmatrix}$$

$$\gamma^0 u_i \bar{u}_i = \begin{pmatrix} (E_i + m)\chi^{(s)}{\chi^{(s)}}^T & -\vec{\sigma}\cdot\vec{p}_i\chi^{(s)}{\chi^{(s)}}^T \\ -\vec{\sigma}\cdot\vec{p}_i\chi^{(s)}{\chi^{(s)}}^T & (E_i - m)\chi^{(s)}{\chi^{(s)}}^T \end{pmatrix}$$

$$\gamma^0 u_f \bar{u}_f = \begin{pmatrix} (E_f + m)\chi^{(s\prime)}{\chi^{(s\prime)}}^T & -\vec{\sigma}\cdot\vec{p}_f\chi^{(s\prime)}{\chi^{(s\prime)}}^T \\ -\vec{\sigma}\cdot\vec{p}_f\chi^{(s\prime)}{\chi^{(s\prime)}}^T & (E_f - m)\chi^{(s\prime)}{\chi^{(s\prime)}}^T \end{pmatrix}$$

- We can see that this will vanish unless $s' = s$ which means that the Coulomb potential does not flip the electron spin when it scatters.

Trace Theorems

- There are easier ways to work with these expressions.
- If we have an un-polarized beam of electrons, then we need to average over the initial spins.
- If we don’t distinguish between the two spin states of the scattered electron then we can sum over the final spins.
- The transition matrix element ensures that the spin won’t flip, so we can’t over-count.

Trace Theorems

$$\sum_s u^{(s)}(p)\bar{u}^{(s)}(p) = \sum_s \begin{pmatrix} (E+m)\chi^{(s)}\chi^{(s)^T} & -\vec{\sigma}\cdot\vec{p}\chi^{(s)}\chi^{(s)^T} \\ \vec{\sigma}\cdot\vec{p}\chi^{(s)}\chi^{(s)^T} & -(E-m)\chi^{(s)}\chi^{(s)^T} \end{pmatrix}$$

$$= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{pmatrix}$$

$$= p^\mu\gamma_\mu + m$$

Likewise,

$$\sum_s v^{(s)}(p)v^{(s)}(p) = p^\mu\gamma_\mu - m$$

Also, *the trace over an odd number of gamma matrices will vanish.*

Coulomb Scattering

- Summing/averaging over the spins:

$$\left|\bar{T}_{fi}\right|^2 = \frac{Z^2 e^4}{2q^4}\,\mathrm{Tr}\left(\gamma^0(\not{p}_f + m)\gamma^0(\not{p}_i + m)\right)$$

$$\frac{Z^2 e^4}{2q^4}\left(Tr\left(\gamma^0\not{p}_f\gamma^0\not{p}_i\right) + 4m^2\right)$$

- Commutation relation: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$\left|\bar{T}_{fi}\right|^2 = \frac{Z^2 e^4}{2q^4}\left(4m^2 - \mathrm{Tr}\left(\not{p}_f\not{p}_i\right) + 8E_iE_f\right)$$

Even More Trace Theorems

$$\mathrm{Tr}(\not{a}\not{b}) = \mathrm{Tr}(2a \cdot b - \not{b}\not{a})$$
$$\mathrm{Tr}(\not{a}\not{b}) = 4a \cdot b$$
$$\left|\bar{T}_{fi}\right|^2 = \frac{Z^2 e^4}{2q^4} \left(4m^2 + 8E_f E_i - 4p_f \cdot p_i\right)$$

- 4-vectors:

$$p_i = (E, \vec{p})$$
$$p_f = (E, \vec{p}')$$
$$p_f \cdot p_i = E^2 - \vec{p} \cdot \vec{p}' = |\vec{p}|^2 + m^2 - |\vec{p}|^2 \cos\theta$$
$$= |\vec{p}|^2(1 - \cos\theta) + m^2 = 2|\vec{p}|^2 \sin^2(\theta/2) + m^2$$
$$\left|\bar{T}_{fi}\right|^2 = \frac{Z^2 e^4}{2q^4} (8E^2 - 8|\vec{p}|^2 \sin^2(\theta/2))$$
$$= \frac{8Z^2 e^4}{2q^4} (|\vec{p}|^2 \cos^2(\theta/2) + m^2)$$
$$q^4 = \left(\left(p_f - p_i\right) \cdot \left(p_f - p_i\right)\right)^2 = \left(2m^2 - 2p_f \cdot p_i\right)^2 = (2m^2 - 2E^2 + 2|\vec{p}|^2 \cos\theta)^2$$
$$= 4|\vec{p}|^4 (1 - \cos\theta)^2 = 8|\vec{p}|^4 \sin^4(\theta/2)$$

Putting it all together…

$$\left|\bar{T}_{fi}\right|^2 = \frac{Z^2 e^4}{2|\vec{p}|^4 \sin^4(\theta/2)} (|\vec{p}|^2 \cos^2(\theta/2) + m^2)$$

- Low energy limit (Rutherford scattering):

$$\frac{d\sigma}{d\Omega} \sim \frac{Z^2 e^4}{|\vec{p}|^4 \sin^4(\theta/2)}$$

Notes

- In this example, the electric field is an external, static entity.
- Momentum is not conserved…
- In reality, the electric field would have a source that would recoil to conserve momentum.
  - The source of the field could be another spin ½ particle
- In quantum field theory, the electromagnetic field can be excited to produce real quanta (photons).