Purdue University Department of Physics

Physics 56400
Introduction to Elementary Particle Physics I

Lecture 14
Fall 2019 Semester

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Dynamics of Particle Interactions

- Schrodinger's equation for a free particle:

$$\hat{H}\Psi = \frac{\hat{p}^2}{2m}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

- In the momentum basis,

$$\langle p|\Psi\rangle = f(p)$$

$$\hat{p}|p\rangle = p|p\rangle$$

$$\frac{p^2}{2m} = \hbar\omega$$

- Time dependence of momentum eigenstates:

$$|p,t\rangle = e^{-i\omega t}|p\rangle$$

$$\omega = \frac{p^2}{2m\hbar} = \frac{E}{\hbar}$$

Dynamics of Particle Interactions

- Try to construct an equation that is consistent with special relativity:

$$\widehat{H}\Psi = \sqrt{\hat{p}^2 + m^2}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

- In the momentum basis,

$$\langle p|\Psi\rangle = f(p)$$

$$\hat{p}|p\rangle = p|p\rangle$$

$$\sqrt{p^2 + m^2} = \hbar\omega$$

- Time dependence of momentum eigenstates:

$$|p, t\rangle = e^{-i\omega t}|p\rangle$$

Dynamics of Particle Interactions

- Try to construct a Hamiltonian that describes spin ½ particles, quantized along the z-axis:

$$\hat{s}_z|+\rangle = \frac{\hbar}{2}|+\rangle$$

$$\hat{s}_z|-\rangle = -\frac{\hbar}{2}|-\rangle$$

- Now, the wavefunction has two components:

$$\langle s_z, p|\Psi\rangle = \begin{pmatrix} \psi_1(p) \\ \psi_2(p) \end{pmatrix}$$

Dynamics of Particle Interactions

- Let's construct a Hamiltonian that is linear in p:

$$\left\langle s_z, p\middle|\hat{H}\middle|\Psi\right\rangle = (\vec{\sigma}\cdot\vec{p} + m)\psi(p)$$

- This is constructed using the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Energy squared:

$$\left\langle s_z, p\middle|\hat{H}^2\middle|\Psi\right\rangle = (\vec{\sigma}\cdot\vec{p} + m)(\vec{\sigma}\cdot\vec{p} + m)\psi(p)$$

- Re-write the dot products using indices:

$$\vec{\sigma}\cdot\vec{p} = \sigma_i p_i$$

$$(\vec{\sigma}\cdot\vec{p})^2 = \sigma_i\sigma_j p_i p_j = \left(\delta_{ij} + i\varepsilon_{ijk}\sigma_k\right)p_i p_j = p^2$$

$$(\vec{\sigma}\cdot\vec{p} + m)(\vec{\sigma}\cdot\vec{p} + m) = (p^2 + m^2 + 2m\vec{\sigma}\cdot\vec{p})$$

- This is inconsistent with $E^2 = p^2 + m^2$ unless $m = 0$.

Dynamics of Particle Interactions

- Try again using a 4-dimensional representation:

$$\left\langle s_z, p \middle| \hat{H} \middle| \Psi \right\rangle = (\vec{\alpha} \cdot \vec{p} + \beta m)\psi(p)$$

- $\vec{\alpha}$ and β are now 4x4 matrices and ψ is a 4-component spinor.

$$\left\langle s_z, p \middle| \hat{H}^2 \middle| \Psi \right\rangle = (\vec{\alpha} \cdot \vec{p} + \beta m)(\vec{\alpha} \cdot \vec{p} + \beta m)\psi(p)$$

- For this to work we must have

$$(\vec{\alpha} \cdot \vec{p})(\vec{\alpha} \cdot \vec{p}) = p^2$$

$$(\vec{\alpha} \cdot \vec{p})\beta + \beta(\vec{\alpha} \cdot \vec{p}) = 0$$

$$\beta^2 = 1 \qquad \text{(4x4 identity matrix)}$$

Dirac Matrices

- This is one representation that works:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(this is not the only such representation)

- Now the cross-terms cancel:

$$\vec{\alpha}\beta + \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = 0$$

Dirac Equation

$$(\vec{\alpha} \cdot \vec{p} - E + \beta m)\psi(p) = 0$$

- We can re-write this in a Lorentz covariant form:

$$\beta(\vec{\alpha} \cdot \vec{p} - E + \beta m)\psi(p) = 0$$

$$\left(\gamma^{\mu} p_{\mu} - m\right)\psi(p) = 0$$

- The gamma matrices are defined

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\gamma} = \beta\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

- Useful notation: $\not{a} = \gamma^{\mu} a_{\mu}$ so we can write the Diract equation like this:

$$(\not{p} - m)\psi(p) = 0$$

Solutions to the Dirac Equation

$$\psi(p) = \begin{pmatrix} u_A(\vec{p}) \\ u_B(\vec{p}) \end{pmatrix}$$

- u_A and u_B are 1x2 column vectors.

$$(\gamma^\mu p_\mu - m)\psi(p) = \begin{pmatrix} E - m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E - m \end{pmatrix} \begin{pmatrix} u_A(\vec{p}) \\ u_B(\vec{p}) \end{pmatrix}$$

- This gives two coupled equations:

$$\vec{\sigma}\cdot\vec{p}\, u_B = (E - m)u_A$$
$$\vec{\sigma}\cdot\vec{p}\, u_A = (E + m)u_B$$

- These can also be written:

$$u_A = \frac{\vec{\sigma}\cdot\vec{p}}{E-m} u_B \qquad\qquad u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} u_A$$

Solutions to the Dirac Equation

- First, suppose that $E > 0$. Then we can pick the basis $u_A = \chi^{(s)}$ where $\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- Then, solutions can be written

$$u^{(s)} = N \begin{pmatrix} \chi^{(s)} \\ \dfrac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^{(s)} \end{pmatrix}$$

- But we can also have $E < 0$ and in this case, let

$$u^{(s+2)} = N \begin{pmatrix} \dfrac{-\vec{\sigma} \cdot \vec{p}}{|E| + m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

Solutions to the Dirac Equation

- Rather than work with $u^{(3)}$ and $u^{(4)}$ which have $E < 0$, it is convenient to introduce the v spinors:

$$v^{(1)}(p) = u^{(4)}(-p)$$
$$v^{(2)}(p) = u^{(3)}(-p)$$

- Then the solutions can be written

$$v^{(s)} = N \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

- The $E < 0$ solutions are interpreted as anti-particles

Solutions to the Dirac Equation

$$(\not{p} - m)u(p) = 0$$ (particles)

$$(-\not{p} - m)u(-p) = 0$$ (anti-particles)

… or …

$$(\not{p} - m)u(p) = 0$$ (particles)

$$(\not{p} + m)v(p) = 0$$ (anti-particles)

Normalization of Solutions

- We will us a Lorentz covariant normalization:

$$\int \psi^\dagger \psi dV = \frac{2E}{V} \qquad\qquad \psi^\dagger \psi = 2E$$

- Positive energy solutions:

$$u^\dagger u = |N|^2 \left(1 + \frac{|\vec{p}|^2}{(E+m)^2}\right)$$

$$= |N|^2 \left(1 + \frac{(E^2 - m^2)}{(E+m)^2}\right)$$

$$= |N|^2 \left(\frac{2E}{E+m}\right) = 2E$$

- Therefore, the normalization is $N = \sqrt{E+m}$.

Solutions to the Dirac Equation

- Positive energy solutions:

$$u^{(s)}(\vec{p}) = \begin{pmatrix} \sqrt{E+m}\,\chi^{(s)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi^{(s)} \end{pmatrix}$$

- Negative energy solutions:

$$v^{(s)}(\vec{p}) = \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi'^{(s)} \\ \sqrt{E+m}\,\chi'^{(s)} \end{pmatrix}$$

$$\chi'^{(1)} = \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \chi'^{(2)} = \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- In both cases, $E = \sqrt{|\vec{p}|^2 + m^2} > 0$.

Position Representation

- We can also use the position basis:

$$\langle x|p\rangle = e^{ixp}$$

$$\langle x|x'\rangle = \int \langle x|p\rangle\langle p|x'\rangle dp = \int e^{ip(x-x')}dp = 2\pi\delta(x-x')$$

- Time dependence:

$$\langle \vec{x}|\vec{p},t\rangle = e^{i(\vec{p}\cdot\vec{x}-Et)} = e^{-ip\cdot x}$$

- In this representation, the momentum and energy operators are

$$\hat{p} = -i\nabla$$

$$\hat{E} = i\partial/\partial t$$

- The Dirac equation can now be written

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x) = 0$$

$$(i\not{\partial} - m)\psi(x) = 0$$

$$\psi(x) = u(p)e^{-ip\cdot x}$$

Adjoint Spinors

- It will be convenient to introduce the adjoint spinors, $\bar{\psi}$ which are defined:

$$\bar{\psi} = \psi^{\dagger}\gamma^{0}$$

- These satisfy the Dirac equation in the adjoint representation:

$$\left[\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi(x)\right]^{\dagger} = 0$$

$$\psi^{\dagger}\left(-i\gamma^{\mu\dagger}\overleftarrow{\partial}_{\mu} - m\right) = 0$$

- But we can use the identities:

$$\gamma^{\mu\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0} \qquad\qquad (\gamma^{0})^{2} = 1$$

$$\bar{\psi}\left(-i\gamma^{\mu}\overleftarrow{\partial}_{\mu} - m\right)\gamma^{0} = 0$$

… or …

$$\bar{\psi}\left(i\overleftarrow{\not\partial} + m\right) = 0$$

Probability Density Currents

- The probability density of a wave function is just

$$\rho = |\psi|^2 = \psi^\dagger \psi$$

- We want to construct a current that satisfies the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

- Observe that we can write

$$\bar{\psi}\left(i\overleftarrow{\partial\!\!\!/} + m\right)\psi + \bar{\psi}\left(i\overrightarrow{\partial\!\!\!/} - m\right)\psi = 0$$

$$i\partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$$

- Therefore, the probability density current is

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

Probability Density Currents

- Consider the positive energy states:

$$\psi^{(s)}(x) = u^{(s)}(p)e^{-ip\cdot x}$$

$$j^{\mu}(x) = \bar{u}^{(s)}\gamma^{\mu}u^{(s)}$$

$$j^0(x) = \begin{pmatrix} \sqrt{E+m}\,\chi^{(s)\dagger} & \chi^{(s)\dagger}\dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}} \end{pmatrix} \begin{pmatrix} \sqrt{E+m}\,\chi^{(s)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi^{(s)} \end{pmatrix}$$

$$= (E+m) + \frac{|\vec{p}|^2}{E+m} = E + m + \frac{E^2 - m^2}{E+m} = 2E$$

$$\vec{j}(x) = \begin{pmatrix} \sqrt{E+m}\,\chi^{(s)\dagger} & \chi^{(s)\dagger}\dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{E+m}\,\chi^{(s)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi^{(s)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{E+m}\,\chi^{(s)\dagger} & \chi^{(s)\dagger}\dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}} \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{E+m}\,\chi^{(s)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi^{(s)} \end{pmatrix}$$

Probability Density Currents

- The current is

$$\vec{j}(x) = \left(\chi^{(s)^\dagger} \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\vec{\sigma} \quad \sqrt{E+m}\,\chi^{(s)^\dagger}\vec{\sigma} \right) \begin{pmatrix} \sqrt{E+m}\,\chi^{(s)} \\ \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi^{(s)} \end{pmatrix}$$

$$= \chi^{(s)^\dagger}\big((\vec{\sigma}\cdot\vec{p})\vec{\sigma} + \vec{\sigma}(\vec{\sigma}\cdot\vec{p})\big)\chi^{(s)}$$

- We can write this as

$$j_j = \chi^{(s)^\dagger}(\sigma_i\sigma_j + \sigma_j\sigma_i)p_i\chi^{(s)}$$

$$= \chi^{(s)^\dagger}\big(\sigma_i\sigma_j + i\varepsilon_{jik}\sigma_k + \delta_{ij}\big)p_i\chi^{(s)}$$

$$\chi^{(s)^\dagger}\big(i\varepsilon_{ijk}\sigma_k - i\varepsilon_{ijk}\sigma_k + 2\delta_{ij}\big)p_i\chi^{(s)} = 2p_j$$

- Thus, $j^\mu(x) = 2p^\mu$ and the probability density flows in the direction of the momentum.

Probability Density Currents

- What about currents for the anti-particles?

$$\psi^{(s+2)}(x) = u^{(s+2)}(-p)e^{ip\cdot x}$$
$$= v^{(s)}(p)e^{ip\cdot x}$$
$$j^{\mu}(x) = \bar{v}^{(s)}\gamma^{\mu}v^{(s)}$$

$$j^0(x) = \begin{pmatrix} {\chi'^{(s)}}^{\dagger}\frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}} & \sqrt{E+m}\,{\chi'^{(s)}}^{\dagger} \end{pmatrix}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi'^{(s)} \\ \sqrt{E+m}\,\chi'^{(s)} \end{pmatrix}$$
$$= E - m + E + m = 2E > 0$$

$$\vec{j}(x) = \begin{pmatrix} {\chi'^{(s)}}^{\dagger}\frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}} & \sqrt{E+m}\,{\chi'^{(s)}}^{\dagger} \end{pmatrix}\begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi'^{(s)} \\ \sqrt{E+m}\,\chi'^{(s)} \end{pmatrix}$$
$$= \begin{pmatrix} \sqrt{E+m}\,{\chi'^{(s)}}^{\dagger}\vec{\sigma} & {\chi'^{(s)}}^{\dagger}\frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\vec{\sigma} \end{pmatrix}\begin{pmatrix} \frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\chi'^{(s)} \\ \sqrt{E+m}\,\chi'^{(s)} \end{pmatrix}$$
$$= {\chi^{(s)}}^{\dagger}(\vec{\sigma}(\vec{\sigma}\cdot\vec{p}) + (\vec{\sigma}\cdot\vec{p})\vec{\sigma})\chi^{(s)}$$
$$= 2\vec{p}$$

Electron Currents

- In general, the probability density current $j^{\mu} = \bar{\psi}\gamma^{\mu}\psi$ describes the motion of electrons with positive energy and momentum $\vec{p}$ as well as electrons with negative energy and momentum $-\vec{p}$.
- If all physical electrons are assigned charge $-e$ then the electric current is
$$j_{EM}^{\mu}(x) = -e\,(\bar{\psi}\gamma^{\mu}\psi)$$
- This naturally obeys the continuity equation:
$$\partial_{\mu} j_{EM}^{\mu}(x) = 0$$

Summary

- We can interpret solutions to the Dirac equation as descriptions of spin ½ particles and their anti-particles.
- Next, we would like to see how to describe the motion of charged fermions in a static (classical) electromagnetic field.
- Then we want to see how they couple to a quantized electromagnetic field.