

Physics 56400
**Introduction to Elementary
Particle Physics I**

Now in PowerPoint!

Lecture 15
Fall 2018 Semester
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- As my field theory instructor once said, this is probably correct modulo factors of i , $\hbar$, and π ...



- But I will at least try...

Interaction with an External Field

- Schrodinger's equation for a free particle:

$$\hat{H}\Psi = \frac{\hat{p}^2}{2m} \Psi$$

- Schrodinger's equation for a particle subjected to a force:

$$\hat{H}\Psi = \left(\frac{\hat{p}^2}{2m} + V \right) \Psi$$
$$\vec{F} = -\nabla V$$

- Hydrogen atom:

$$V(r) = -\frac{Ze^2}{r}$$
$$F_r = -\frac{\partial V}{\partial r} = -\frac{Ze^2}{r^2}$$
$$\hat{H}\Psi = \left(\frac{\hat{p}^2}{2m} - \frac{Ze^2}{r} \right) \Psi$$

Interaction with an External Field

- Minimal substitution:

$$\begin{aligned} p^\mu &\rightarrow p^\mu - qA^\mu \\ &= p^\mu + eA^\mu \quad (\text{for an electron with } q = -e) \end{aligned}$$

- Vector potential,

$$A^\mu = (\phi, \vec{A})$$

- Hydrogen atom:

$$\phi = \frac{Ze}{r}$$

- Schrodinger's equation:

$$\hat{H}\Psi = \left(\frac{\hat{p}^2}{2m} + e\phi \right) \Psi = \left(\frac{\hat{p}^2}{2m} - \frac{Ze^2}{r} \right) \Psi$$

Interaction with an External Field

- Dirac equation (homogeneous):

$$(\gamma^\mu p_\mu - m)\psi(p) = 0$$

- Minimal substitution:

$$p^\mu \rightarrow p^\mu + eA^\mu$$

- Dirac equation with an external field:

$$\begin{aligned}(\gamma^\mu p_\mu - m)\psi(p) &= -e\gamma^\mu A_\mu\psi(p) \\ &= \gamma^0 V\psi(p)\end{aligned}$$

- We can solve this equation using perturbation theory by expanding in powers of e .

Time Dependent Perturbation Theory

- Suppose that $|i\rangle$ is an eigenstate of a “free” Hamiltonian, $\hat{H}_0$:

$$\hat{H}_0|i\rangle = E_i|i\rangle$$

- The time evolution operator gives the state at a later time:

$$|i, t\rangle = U(t)|i\rangle = e^{-i\hat{H}_0 t}|i\rangle = e^{-iE_i t}|i\rangle$$

- For an unstable particle with decay rate Γ , the time evolution operator can be written

$$U(t) = e^{-i\hat{H}_0 t - \Gamma t/2}$$

- The part that is responsible for the decay is written as a perturbation to the free Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}$$

Time Dependent Perturbation Theory

- We want to find an expression for the state at time t given that it was in state $|i\rangle$ at time $t = 0$.
- Write this in terms of free particle states at time t :

$$|f, t\rangle = e^{-iE_f t} |f\rangle$$

$$|i, t\rangle = \sum_n |n, t\rangle \langle n, t | i, t\rangle$$

$$= \sum_n |n, t\rangle \langle n, t | U(t) | i\rangle$$

$$= \sum_n c_n(t) |n, t\rangle$$

Time Dependent Perturbation Theory

$$c_n(t) = \langle n, t | U(t) | i \rangle$$

- This is written in terms of free particle states so

$$\langle n, t | = e^{iE_n t} \langle n |$$

- The time evolution operator is written using the full Hamiltonian:

$$U(t) = e^{-i\hat{H}t} = e^{-i(\hat{H}_0 + \hat{V})t}$$

- Thus,

$$\begin{aligned} \frac{dc_n}{dt} &= iE_n \langle n, t | U(t) | i \rangle - i \langle n, t | (\hat{H}_0 + \hat{V}) U(t) | i \rangle \\ &= -i \langle n, t | \hat{V} U(t) | i \rangle = -i \langle n, t | \hat{V} | i, t \rangle \\ &= - \sum_m i \langle n, t | \hat{V} | m, t \rangle \langle m, t | i, t \rangle = - \sum_m i \langle n, t | \hat{V} | m, t \rangle c_m(t) \end{aligned}$$

Time Dependent Perturbation Theory

- When $t \rightarrow 0$ we expect that $c_i(t) \rightarrow e^{-iE_it - \Gamma t/2}$ and $c_j(t) \rightarrow 0$ when $i \neq j$

$$\frac{dc_n}{dt} = -ie^{i(E_n - E_i)t} e^{-\Gamma t/2} \langle n | \hat{V} | i \rangle$$

$$c_n(t) = -i \int_0^t e^{i(E_n - E_i)t'} e^{-\frac{\Gamma t'}{2}} \langle n | \hat{V} | i \rangle dt'$$

- But, if $\langle n | \hat{V} | i \rangle = V_{ni}$ does not depend too strongly on energy then we can take it out of the integral:

$$\begin{aligned} c_n(t) &= -iV_{ni} \int_0^t e^{i(E_n - E_i)t'} e^{-\frac{\Gamma t'}{2}} \\ &= -i \frac{V_{ni}}{i(E_n - E_i) - \Gamma/2} e^{i(E_n - E_i)t} e^{-\frac{\Gamma t}{2}} \Big|_0^t \end{aligned}$$

Time Dependent Perturbation Theory

- What we really care about is the asymptotic limit as $t \rightarrow \infty$

$$c_n(\infty) = i \frac{V_{ni}}{i(E_n - E_i) - \Gamma/2}$$

- The probability of observing the final state $|f\rangle$ at some time in the distant future is

$$\mathcal{P}(i \rightarrow f) = |c_f(\infty)|^2 = \frac{|V_{fi}|^2}{(E_f - E_i)^2 + (\Gamma/2)^2}$$

Time Dependent Perturbation Theory

- In practice, we are usually interested in the probability of observing a particular class of final states.
- We need to sum over all relevant kinematic configurations of final states, f :

$$\mathcal{P}(i \rightarrow f) = \sum_f |c_f(\infty)|^2 = \int_0^\infty \frac{\rho_f(E) |V_{fi}|^2}{(E - E_i)^2 + (\Gamma/2)^2} dE$$

- $\rho_f(E)$ is the density of final states with energy E .

Time Dependent Perturbation Theory

- The integrand is very small except when $E \approx E_i$ so

$$\begin{aligned}\mathcal{P}(i \rightarrow f) &= \rho_f(E) |V_{fi}|^2 \int_{-\infty}^{\infty} \frac{dE}{(E - E_i)^2 + (\Gamma/2)^2} \\ &= \rho_f(E) |V_{fi}|^2 \cdot \frac{2\pi}{\Gamma}\end{aligned}$$

- If we sum over all possible final states we expect that

$$\sum_f \mathcal{P}(i \rightarrow f) = 1$$

- The individual probabilities are the branching fractions with partial widths

$$\Gamma_f = \Gamma \text{Br}(i \rightarrow f)$$

- The total decay rate is

$$\Gamma = \sum_f \Gamma_f = \sum_f 2\pi \rho_f(E) |V_{fi}|^2$$

Density of States

- How many states are there with an energy between E and $E + dE$?
- Consider a particle confined to a box with sides of length L .

$$\phi(\vec{x}) = \langle \vec{x} | \phi \rangle = N e^{i\vec{k} \cdot \vec{x}}$$

$$\phi(\vec{x}, t) = N e^{i\vec{k} \cdot \vec{x} - iEt}$$

- Periodic boundary conditions:

$$k_x = 2\pi n_x / L$$

$$k_y = 2\pi n_y / L$$

$$k_z = 2\pi n_z / L$$

Density of States

$$\Delta n_x \Delta n_y \Delta n_z = \frac{\Delta k_x \Delta k_y \Delta k_z}{(2\pi)^3} L^3$$

- Probability density is

$$\rho = i \left(\phi^*(x) \frac{\partial \phi}{\partial t} - \phi(x) \frac{\partial \phi^*}{\partial t} \right) = 2E |N|^2$$

- We integrate this over the whole box:

$$\int d^3x \rho(x) = 2E |N|^2 L^3$$

- A suitable normalization would be

$$N = 1/\sqrt{V}$$

Density of States

- The number of states per volume is $d^3k L^3 / (2\pi)^3$
- Particles per volume is $2E$
- Number of final states per particle is $\frac{d^3k}{(2\pi)^3} \cdot \frac{L^3}{2E}$
- In general,

$$\rho_f = \frac{d^3p_1 L^3}{(2\pi)^3 2E_1} \cdot \frac{d^3p_2 L^3}{(2\pi)^3 2E_2} \cdots \delta(E_f - E_i)$$

- The factors of L^3 will cancel with the normalization of states in the matrix element $\langle f|V|i\rangle$.

Transition Amplitude

- The energy-conserving delta function and V_{fi} can be combined into the “transition matrix element”:

$$T_{fi} = -i \langle f | \hat{V} | i \rangle \int e^{i(E_f - E_i)t} dt$$

- In the position representation:

$$T_{fi} = -i \int d^3x d^3x' \langle f | x' \rangle \langle x' | \hat{V} | x \rangle \langle x | i \rangle \int e^{i(E_f - E_i)t} dt$$

- If, as is often the case, the operator $\hat{V}$ is local, then

$$\langle x' | \hat{V} | x \rangle = \delta(x' - x) \langle x | \hat{V} | x \rangle$$

- The transition matrix element can be written

$$T_{fi} = -i \int d^3x \phi_f^*(\vec{x}) V(\vec{x}) \phi_i(\vec{x}) \int e^{i(E_f - E_i)t} dt = -i \int d^4x \phi_f^*(x) V(x) \phi_i(x)$$

Cross Sections and Decay Rates

- Differential decay rate:

$$d\Gamma = \frac{2\pi}{M} \prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} |\langle f|V|i\rangle|^2$$

- Differential cross section:

$$d\sigma = \frac{2\pi}{|v_{AB}| \cdot 2E_A \cdot 2E_B} \prod_i \frac{d^3 p_i}{(2\pi)^3 2E_i} |\langle f|V|i\rangle|^2$$

Transition Amplitude for Spin ½ Particles

- The free particle solutions to the Dirac equation are

$$\psi(x) = u(p)e^{-ip \cdot x}$$

- The free particle solutions to the adjoint Dirac equation are

$$\bar{\psi}(x) = \bar{u}(p)e^{ip \cdot x}$$

- The perturbation to the Hamiltonian is

$$V(x) = e\gamma^0\gamma^\mu A_\mu(x)$$

- The transition matrix element is

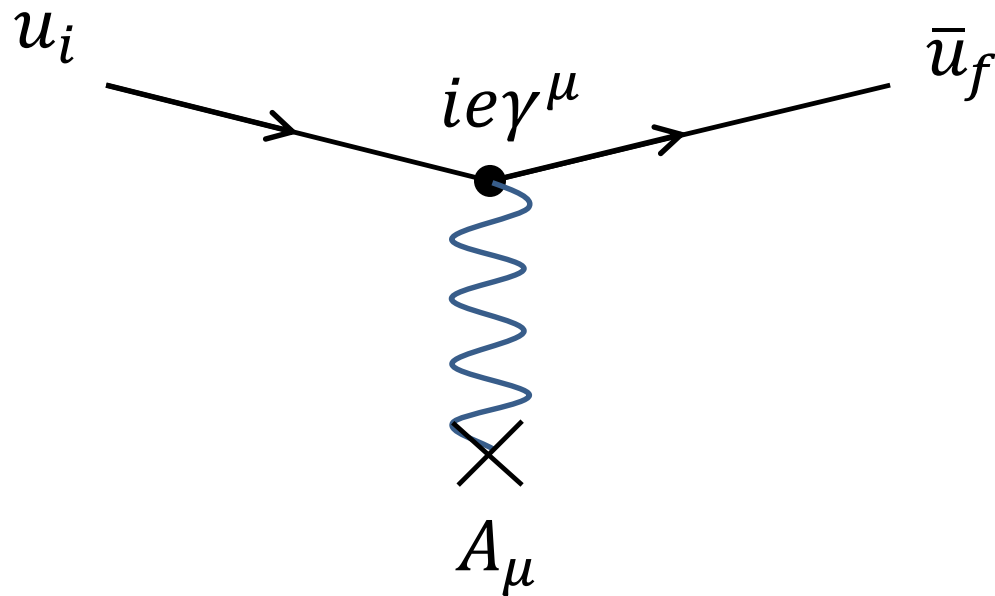
$$T_{fi} = -ie \int d^4x \left(\bar{\psi}_f(x) \gamma^\mu \psi_i(x) \right) A_\mu(x) e^{ix \cdot (p_f - p_i)}$$

$$= -ie \bar{u}(p_f) \gamma^\mu u(p_i) A_\mu(q)$$

$$q = p_f - p_i$$

Transition Amplitudes for Spin ½ Particles

- We can describe the form of this expression graphically:



Coulomb Scattering

- Consider scattering from a static central potential:

$$A^\mu(x) = (\phi(x), \vec{0})$$

$$\phi(x) = \frac{Ze}{r}$$

- Fourier transform:

$$A^0(q) = \frac{Ze}{q^2}$$

- Transition matrix element:

$$T_{fi} = \frac{Ze^2 \bar{u}_f \gamma^0 u_i}{q^2}$$

Coulomb Scattering

- To calculate the probability we need $|T_{fi}|^2$

$$\begin{aligned}|T_{fi}|^2 &= \frac{Z^2 e^4}{q^4} (\bar{u}_f \gamma^0 u_i)^\dagger (\bar{u}_f \gamma^0 u_i) \\ &= \frac{Z^2 e^4}{q^4} (\bar{u}_i \gamma^0 u_f) (\bar{u}_f \gamma^0 u_i) \\ &= \frac{Z^2 e^4}{q^4} \text{Tr}(\gamma^0 u_f \bar{u}_f \gamma^0 u_i \bar{u}_i)\end{aligned}$$

Coulomb Scattering

- We can try to evaluate this explicitly:

$$u_i \bar{u}_i = \begin{pmatrix} (E_i + m) \chi^{(s)} \chi^{(s)T} & -\vec{\sigma} \cdot \vec{p}_i \chi^{(s)} \chi^{(s)T} \\ \vec{\sigma} \cdot \vec{p}_i \chi^{(s)} \chi^{(s)T} & -(E_i - m) \chi^{(s)} \chi^{(s)T} \end{pmatrix}$$

$$\gamma^0 u_i \bar{u}_i = \begin{pmatrix} (E_i + m) \chi^{(s)} \chi^{(s)T} & -\vec{\sigma} \cdot \vec{p}_i \chi^{(s)} \chi^{(s)T} \\ -\vec{\sigma} \cdot \vec{p}_i \chi^{(s)} \chi^{(s)T} & (E_i - m) \chi^{(s)} \chi^{(s)T} \end{pmatrix}$$

$$\gamma^0 u_f \bar{u}_f = \begin{pmatrix} (E_f + m) \chi^{(s')} \chi^{(s')T} & -\vec{\sigma} \cdot \vec{p}_f \chi^{(s')} \chi^{(s')T} \\ -\vec{\sigma} \cdot \vec{p}_f \chi^{(s')} \chi^{(s')T} & (E_f - m) \chi^{(s')} \chi^{(s')T} \end{pmatrix}$$

- We can see that this will vanish unless $s' = s$ which means that the Coulomb potential does not flip the electron spin when it scatters.

Trace Theorems

- There are easier ways to work with these expressions.
- If we have an un-polarized beam of electrons, then we need to average over the initial spins.
- If we don't distinguish between the two spin states of the scattered electron then we can sum over the final spins.
- The transition matrix element ensures that the spin won't flip, so we can't over-count.

Trace Theorems

$$\begin{aligned}\sum_s u^{(s)}(p)\bar{u}^{(s)}(p) &= \sum_s \begin{pmatrix} (E+m)\chi^{(s)}\chi^{(s)T} & -\vec{\sigma}\cdot\vec{p}\chi^{(s)}\chi^{(s)T} \\ \vec{\sigma}\cdot\vec{p}\chi^{(s)}\chi^{(s)T} & -(E-m)\chi^{(s)}\chi^{(s)T} \end{pmatrix} \\ &= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{pmatrix} \\ &= p^\mu\gamma_\mu + m\end{aligned}$$

Likewise,

$$\sum_s v^{(s)}(p)\bar{v}^{(s)}(p) = p^\mu\gamma_\mu - m$$

Also, the trace over an odd number of gamma matrices will vanish.

Coulomb Scattering

- Summing/averaging over the spins:

$$|\bar{T}_{fi}|^2 = \frac{Z^2 e^4}{2q^4} \text{Tr} \left(\gamma^0 (\not{p}_f + m) \gamma^0 (\not{p}_i + m) \right)$$

$$\frac{Z^2 e^4}{2q^4} \left(\text{Tr}(\gamma^0 \not{p}_f \gamma^0 \not{p}_i) + 4m^2 \right)$$

- Commutation relation: $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$

$$|\bar{T}_{fi}|^2 = \frac{Z^2 e^4}{2q^4} \left(4m^2 - \text{Tr}(\not{p}_f \not{p}_i) + 8E_i E_f \right)$$

Even More Trace Theorems

$$\text{Tr}(\cancel{a}\cancel{b}) = \text{Tr}(2a \cdot b - \cancel{b}\cancel{a})$$

$$\text{Tr}(\cancel{a}\cancel{b}) = 4a \cdot b$$

$$|\bar{T}_{fi}|^2 = \frac{Z^2 e^4}{2q^4} (4m^2 + 8E_f E_i - 4p_f \cdot p_i)$$

- 4-vectors:

$$p_i = (E, \vec{p})$$

$$p_f = (E, \vec{p}')$$

$$\begin{aligned} p_f \cdot p_i &= E^2 - \vec{p} \cdot \vec{p}' = |\vec{p}|^2 + m^2 - |\vec{p}|^2 \cos \theta \\ &= |\vec{p}|^2 (1 - \cos \theta) + m^2 = 2|\vec{p}|^2 \sin^2(\theta/2) + m^2 \end{aligned}$$

$$|\bar{T}_{fi}|^2 = \frac{Z^2 e^4}{2q^4} (8E^2 - 8|\vec{p}|^2 \sin^2(\theta/2))$$

$$= \frac{8Z^2 e^4}{2q^4} (|\vec{p}|^2 \cos^2(\theta/2) + m^2)$$

$$\begin{aligned} q^4 &= \left((p_f - p_i) \cdot (p_f - p_i) \right)^2 = (2m^2 - 2p_f \cdot p_i)^2 = (2m^2 - 2E^2 + 2|\vec{p}|^2 \cos \theta)^2 \\ &= 4|\vec{p}|^4 (1 - \cos \theta)^2 = 8|\vec{p}|^4 \sin^4(\theta/2) \end{aligned}$$

Putting it all together...

$$|\bar{T}_{fi}|^2 = \frac{Z^2 e^4}{2|\vec{p}|^4 \sin^4(\theta/2)} (|\vec{p}|^2 \cos^2(\theta/2) + m^2)$$

- Low energy limit (Rutherford scattering):

$$\frac{d\sigma}{d\Omega} \sim \frac{Z^2 e^4}{|\vec{p}|^4 \sin^4(\theta/2)}$$

Notes

- In this example, the electric field is an external, static entity.
- Momentum is not conserved...
- In reality, the electric field would have a source that would recoil to conserve momentum.
 - The source of the field could be another spin $\frac{1}{2}$ particle
- In quantum field theory, the electromagnetic field can be excited to produce real quanta (photons).