

Physics 56400

Introduction to Elementary Particle Physics I

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Lecture 14
Fall 2018 Semester
Prof. Matthew Jones

Dynamics of Particle Interactions

- Schrodinger's equation for a free particle:

$$\hat{H}\Psi = \frac{\hat{p}^2}{2m}\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

- In the momentum basis,

$$\langle p|\Psi\rangle = f(p)$$

$$\hat{p}|p\rangle = p|p\rangle$$

$$\frac{p^2}{2m} = \hbar\omega$$

- Time dependence of momentum eigenstates:

$$|p, t\rangle = e^{-i\omega t}|p\rangle$$

$$\omega = \frac{p^2}{2m\hbar} = \frac{E}{\hbar}$$

Dynamics of Particle Interactions

- Try to construct an equation that is consistent with special relativity:

$$\hat{H}\Psi = \sqrt{\hat{p}^2 + m^2}\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

- In the momentum basis,

$$\langle p|\Psi\rangle = f(p)$$

$$\hat{p}|p\rangle = p|p\rangle$$

$$\sqrt{p^2 + m^2} = \hbar\omega$$

- Time dependence of momentum eigenstates:

$$|p, t\rangle = e^{-i\omega t}|p\rangle$$

Dynamics of Particle Interactions

- Try to construct a Hamiltonian that describes spin $\frac{1}{2}$ particles, quantized along the z-axis:

$$\hat{s}_z |+\rangle = \frac{\hbar}{2} |+\rangle$$
$$\hat{s}_z |-\rangle = -\frac{\hbar}{2} |-\rangle$$

- Now, the wavefunction has two components:

$$\langle s_z, p | \Psi \rangle = \begin{pmatrix} \psi_1(p) \\ \psi_2(p) \end{pmatrix}$$

Dynamics of Particle Interactions

- Let's construct a Hamiltonian that is linear in p :

$$\langle s_z, p | \hat{H} | \Psi \rangle = (\vec{\sigma} \cdot \vec{p} + m) \psi(p)$$

- This is constructed using the Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- Energy squared:

$$\langle s_z, p | \hat{H}^2 | \Psi \rangle = (\vec{\sigma} \cdot \vec{p} + m)(\vec{\sigma} \cdot \vec{p} + m) \psi(p)$$

- Re-write the dot products using indices:

$$\vec{\sigma} \cdot \vec{p} = \sigma_i p_i$$

$$(\vec{\sigma} \cdot \vec{p})^2 = \sigma_i \sigma_j p_i p_j = (\delta_{ij} + i \varepsilon_{ijk} \sigma_k) p_i p_j = p^2$$

$$(\vec{\sigma} \cdot \vec{p} + m)(\vec{\sigma} \cdot \vec{p} + m) = (p^2 + m^2 + 2m\vec{\sigma} \cdot \vec{p})$$

- This is inconsistent with $E^2 = p^2 + m^2$ unless $m = 0$.

Dynamics of Particle Interactions

- Try again using a 4-dimensional representation:

$$\langle s_z, p | \hat{H} | \Psi \rangle = (\vec{\alpha} \cdot \vec{p} + \beta m) \psi(p)$$

- $\vec{\alpha}$ and β are now 4x4 matrices and ψ is a 4-component spinor.

$$\langle s_z, p | \hat{H}^2 | \Psi \rangle = (\vec{\alpha} \cdot \vec{p} + \beta m)(\vec{\alpha} \cdot \vec{p} + \beta m) \psi(p)$$

- For this to work we must have

$$\begin{aligned}(\vec{\alpha} \cdot \vec{p})(\vec{\alpha} \cdot \vec{p}) &= p^2 \\(\vec{\alpha} \cdot \vec{p})\beta + \beta(\vec{\alpha} \cdot \vec{p}) &= 0\end{aligned}$$

$$\beta^2 = 1 \quad (4 \times 4 \text{ identity matrix})$$

Dirac Matrices

- This is one representation that works:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(this is not the only such representation)

- Now the cross-terms cancel:

$$\begin{aligned} \vec{\alpha}\beta + \beta\vec{\alpha} &= \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -\vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} = 0 \end{aligned}$$

Dirac Equation

$$(\vec{\alpha} \cdot \vec{p} - E + \beta m)\psi(p) = 0$$

- We can re-write this in a Lorentz covariant form:

$$\beta(\vec{\alpha} \cdot \vec{p} - E + \beta m)\psi(p) = 0$$

$$(\gamma^\mu p_\mu - m)\psi(p) = 0$$

- The gamma matrices are defined

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{\gamma} = \beta \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}$$

- Useful notation: $\not{a} = \gamma^\mu a_\mu$ so we can write the Dirac equation like this:

$$(\not{p} - m)\psi(p) = 0$$

Solutions to the Dirac Equation

$$\psi(p) = \begin{pmatrix} u_A(\vec{p}) \\ u_B(\vec{p}) \end{pmatrix}$$

- u_A and u_B are 1x2 column vectors.

$$(\gamma^\mu p_\mu - m)\psi(p) = \begin{pmatrix} E - m & -\vec{\sigma} \cdot \vec{p} \\ \vec{\sigma} \cdot \vec{p} & -E - m \end{pmatrix} \begin{pmatrix} u_A(\vec{p}) \\ u_B(\vec{p}) \end{pmatrix}$$

- This gives two coupled equations:

$$\vec{\sigma} \cdot \vec{p} u_B = (E - m)u_A$$

$$\vec{\sigma} \cdot \vec{p} u_A = (E + m)u_B$$

- These can also be written:

$$u_A = \frac{\vec{\sigma} \cdot \vec{p}}{E - m} u_B$$

$$u_B = \frac{\vec{\sigma} \cdot \vec{p}}{E + m} u_A$$

Solutions to the Dirac Equation

- First, suppose that $E > 0$. Then we can pick the basis $u_A = \chi^{(s)}$ where $\chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
- Then, solutions can be written

$$u^{(s)} = N \begin{pmatrix} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^{(s)} \end{pmatrix}$$

- But we can also have $E < 0$ and in this case, let

$$u^{(s+2)} = N \begin{pmatrix} \frac{-\vec{\sigma} \cdot \vec{p}}{|E| + m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

Solutions to the Dirac Equation

- Rather than work with $u^{(3)}$ and $u^{(4)}$ which have $E < 0$, it is convenient to introduce the v spinors:

$$v^{(1)}(p) = u^{(4)}(-p)$$

$$v^{(2)}(p) = u^{(3)}(-p)$$

- Then the solutions can be written

$$v^{(s)} = N \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{E + m} \chi^{(s)} \\ \chi^{(s)} \end{pmatrix}$$

- The $E < 0$ solutions are interpreted as anti-particles

Solutions to the Dirac Equation

$$\begin{aligned}(\not{p} - m)u(p) &= 0 && \text{(particles)} \\ (-\not{p} - m)u(-p) &= 0 && \text{(anti-particles)}\end{aligned}$$

... or ...

$$\begin{aligned}(\not{p} - m)u(p) &= 0 && \text{(particles)} \\ (\not{p} + m)v(p) &= 0 && \text{(anti-particles)}\end{aligned}$$

Normalization of Solutions

- We will use a Lorentz covariant normalization:

$$\int \psi^\dagger \psi dV = \frac{2E}{V} \quad \psi^\dagger \psi = 2E$$

- Positive energy solutions:

$$\begin{aligned} u^\dagger u &= |N|^2 \left(1 + \frac{|\vec{p}|^2}{(E + m)^2} \right) \\ &= |N|^2 \left(1 + \frac{(E^2 - m^2)}{(E + m)^2} \right) \\ &= |N|^2 \left(\frac{2E}{E + m} \right) = 2E \end{aligned}$$

- Therefore, the normalization is $N = \sqrt{E + m}$.

Solutions to the Dirac Equation

- Positive energy solutions:

$$u^{(s)}(\vec{p}) = \begin{pmatrix} \sqrt{E + m} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E + m}} \chi^{(s)} \end{pmatrix}$$

- Negative energy solutions:

$$v^{(s)}(\vec{p}) = \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E + m}} \chi'^{(s)} \\ \sqrt{E + m} \chi'^{(s)} \end{pmatrix}$$

$$\chi'^{(1)} = \chi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \chi'^{(2)} = \chi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

- In both cases, $E = \sqrt{|\vec{p}|^2 + m^2} > 0$.

Position Representation

- We can also use the position basis:

$$\langle x|p\rangle = e^{ixp}$$

$$\langle x|x'\rangle = \int \langle x|p\rangle \langle p|x'\rangle dp = \int e^{ip(x-x')} dp = 2\pi\delta(x-x')$$

- Time dependence:

$$\langle \vec{x}|\vec{p}, t\rangle = e^{i(\vec{p}\cdot\vec{x}-Et)} = e^{-ip\cdot x}$$

- In this representation, the momentum and energy operators are

$$\hat{p} = -i\nabla$$

$$\hat{E} = i\partial/\partial t$$

- The Dirac equation can now be written

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0$$

$$(i\not{\partial} - m)\psi(x) = 0$$

$$\psi(x) = u(p)e^{-ip\cdot x}$$

Adjoint Spinors

- It will be convenient to introduce the adjoint spinors, $\bar{\psi}$ which are defined:

$$\bar{\psi} = \psi^\dagger \gamma^0$$

- These satisfy the Dirac equation in the adjoint representation:

$$[(i\gamma^\mu \partial_\mu - m)\psi(x)]^\dagger = 0$$

$$\psi^\dagger (-i\gamma^{\mu\dagger} \overleftarrow{\partial}_\mu - m) = 0$$

- But we can use the identities:

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0 \qquad (\gamma^0)^2 = 1$$

$$\bar{\psi} (-i\gamma^\mu \overleftarrow{\partial}_\mu - m) \gamma^0 = 0$$

... or ...

$$\bar{\psi} (i\overleftarrow{\not{\partial}} + m) = 0$$

Probability Density Currents

- The probability density of a wave function is just

$$\rho = |\psi|^2 = \psi^\dagger \psi$$

- We want to construct a current that satisfies the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0$$

- Observe that we can write

$$\begin{aligned} \bar{\psi}(i\vec{\not{\partial}} + m)\psi + \bar{\psi}(i\vec{\not{\partial}} - m)\psi &= 0 \\ i\partial_\mu(\bar{\psi}\gamma^\mu\psi) &= 0 \end{aligned}$$

- Therefore, the probability density current is

$$j^\mu = \bar{\psi}\gamma^\mu\psi$$

Probability Density Currents

- Consider the positive energy states:

$$\psi^{(s)}(x) = u^{(s)}(p)e^{-ip \cdot x}$$

$$j^\mu(x) = \bar{u}^{(s)} \gamma^\mu u^{(s)}$$

$$j^0(x) = \begin{pmatrix} \sqrt{E+m} \chi^{(s)\dagger} & \chi^{(s)\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \end{pmatrix} \begin{pmatrix} \sqrt{E+m} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \chi^{(s)} \end{pmatrix}$$

$$= (E+m) + \frac{|\vec{p}|^2}{E+m} = E+m + \frac{E^2 - m^2}{E+m} = 2E$$

$$\vec{j}(x) = \begin{pmatrix} \sqrt{E+m} \chi^{(s)\dagger} & \chi^{(s)\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{E+m} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \chi^{(s)} \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{E+m} \chi^{(s)\dagger} & \chi^{(s)\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \end{pmatrix} \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \sqrt{E+m} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \chi^{(s)} \end{pmatrix}$$

Probability Density Currents

- The current is

$$\vec{j}(x) = \begin{pmatrix} \chi^{(s)\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \vec{\sigma} & \sqrt{E+m} \chi^{(s)\dagger} \vec{\sigma} \end{pmatrix} \begin{pmatrix} \sqrt{E+m} \chi^{(s)} \\ \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \chi^{(s)} \end{pmatrix}$$

$$= \chi^{(s)\dagger} ((\vec{\sigma} \cdot \vec{p}) \vec{\sigma} + \vec{\sigma} (\vec{\sigma} \cdot \vec{p})) \chi^{(s)}$$

- We can write this as

$$j_j = \chi^{(s)\dagger} (\sigma_i \sigma_j + \sigma_j \sigma_i) p_i \chi^{(s)}$$

$$= \chi^{(s)\dagger} (\sigma_i \sigma_j + i \varepsilon_{jik} \sigma_k + \delta_{ij}) p_i \chi^{(s)}$$

$$\chi^{(s)\dagger} (i \varepsilon_{ijk} \sigma_k - i \varepsilon_{ijk} \sigma_k + 2 \delta_{ij}) p_i \chi^{(s)} = 2 p_j$$

- Thus, $j^\mu(x) = 2p^\mu$ and the probability density flows in the direction of the momentum.

Probability Density Currents

- What about currents for the anti-particles?

$$\begin{aligned}\psi^{(s+2)}(x) &= u^{(s+2)}(-p)e^{ip \cdot x} \\ &= v^{(s)}(p)e^{ip \cdot x} \\ j^\mu(x) &= \bar{v}^{(s)}\gamma^\mu v^{(s)}\end{aligned}$$

$$\begin{aligned}j^0(x) &= \left(\chi'^{(s)\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \quad \sqrt{E+m} \chi'^{(s)\dagger} \right) \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \chi'^{(s)} \\ \sqrt{E+m} \chi'^{(s)} \end{pmatrix} \\ &= E - m + E + m = 2E > 0\end{aligned}$$

$$\begin{aligned}\vec{j}(x) &= \left(\chi'^{(s)\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \quad \sqrt{E+m} \chi'^{(s)\dagger} \right) \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \chi'^{(s)} \\ \sqrt{E+m} \chi'^{(s)} \end{pmatrix} \\ &= \left(\sqrt{E+m} \chi'^{(s)\dagger} \vec{\sigma} \quad \chi'^{(s)\dagger} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \vec{\sigma} \right) \begin{pmatrix} \frac{\vec{\sigma} \cdot \vec{p}}{\sqrt{E+m}} \chi'^{(s)} \\ \sqrt{E+m} \chi'^{(s)} \end{pmatrix} \\ &= \chi^{(s)\dagger} (\vec{\sigma}(\vec{\sigma} \cdot \vec{p}) + (\vec{\sigma} \cdot \vec{p})\vec{\sigma}) \chi^{(s)} \\ &= 2\vec{p}\end{aligned}$$

Electron Currents

- In general, the probability density current $j^\mu = \bar{\psi}\gamma^\mu\psi$ describes the motion of electrons with positive energy and momentum $\vec{p}$ as well as electrons with negative energy and momentum $-\vec{p}$.
- If all physical electrons are assigned charge $-e$ then the electric current is

$$j_{EM}^\mu(x) = -e (\bar{\psi}\gamma^\mu\psi)$$

- This naturally obeys the continuity equation:

$$\partial_\mu j_{EM}^\mu(x) = 0$$

Summary

- We can interpret solutions to the Dirac equation as descriptions of spin $\frac{1}{2}$ particles and their anti-particles.
- Next, we would like to see how to describe the motion of charged fermions in a static (classical) electromagnetic field.
- Then we want to see how they couple to a quantized electromagnetic field.