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Decay rate.

$$\Gamma = \frac{2\pi}{\hbar} \rho_f(E_i) \left| \langle f | V | i \rangle \right|^2$$

we'll start dropping factors of $\hbar$. ie, use $\hbar = 1$.

where it is implied that we have already summed over all "equivalent" final states.

We argued that $\rho_f(E_i)$ was of the form

$$\prod_i \frac{d^3 p_i \, L^3}{(2\pi)^3 \cdot 2E_i}$$

where the product is over all final state particles. Each term is the number of final states that particle i has when its momentum is between $\vec{p}$ and $\vec{p} + d\vec{p}$.

When confined to a box of volume L^3 we wrote $\phi(x) = \dfrac{e^{i\vec{p}\cdot\vec{x}}}{L^{3/2}}$ so that the probability density was $\rho = \dfrac{2E}{L^3}$ and the total number of particles in the box was $\int d^3x \, \rho = 2E$.

$= 2M$ when at rest

$$\text{states per box} = \frac{d^3p \, L^3}{(2\pi)^3}$$

$$\text{particles per box} = 2E$$

$$\text{states per particle} = \frac{d^3p \, L^3}{(2\pi)^3 \cdot 2E}$$

Now consider the matrix element, $\langle f|V|i\rangle$. In the position representation this is

$$\langle f|V|i\rangle = \int d^3x'\, d^3x\, \langle f|\vec{x}'\rangle \langle \vec{x}'|V|\vec{x}\rangle \langle \vec{x}|i\rangle$$

$$= \int d^3x'\, d^3x\, \phi_f^*(\vec{x}')\, \langle \vec{x}'|V|\vec{x}\rangle\, \phi_i(\vec{x})$$

But $\langle \vec{x}'|V|\vec{x}\rangle$ must be of the form

$$\langle \vec{x}'|V|\vec{x}\rangle = \langle \vec{x}'|V|\vec{x}\rangle\, \delta^3(\vec{x}'-\vec{x})$$

or else a state at position $\vec{x}$ could influence another state at $\vec{x}' \neq \vec{x}$ violating locality.

So,

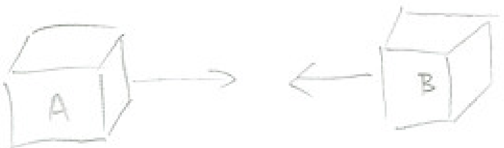
$$\langle f|V|i\rangle = \int d^3x\, \phi_f^*(\vec{x})\, V(\vec{x})\, \phi_i(\vec{x})$$

$$= \int d^3x\, \underbrace{\frac{e^{-i\vec{p}_f\cdot\vec{x}'}}{L^{3/2}}}_{}\, V(\vec{x})\, \frac{e^{i\vec{p}_i\cdot\vec{x}}}{L^{3/2}}$$

or $\dfrac{e^{i(\sum_n \vec{p})\cdot\vec{x}}}{(L^{3/2})^n}$ when there are n particles in the final state.

Cross sections

Consider a box containing one particle colliding with a box containing another particle:



with relative velocity v.

Reaction rate, $R = \mathcal{L}\sigma$

$\mathcal{L}$ is the luminosity: the number of beam particles per unit area per unit time.

σ is the cross section: the "size" of a target particle. Low probability of interaction is equivalent to small size.

# of beam particles per unit area per unit time is $|v_A| \cdot \frac{2E_A}{L^3}$

# of target particles per unit volume is $\frac{2E_B}{L^3}$

$$\mathcal{L} = |v_A| \cdot \frac{2E_A}{L^3} \cdot \frac{2E_B}{L^3}$$

So $\sigma = \frac{R}{\mathcal{L}} = \frac{1}{\mathcal{L}} \frac{2\pi}{\hbar} \rho_f(E_i) |\langle f|V|i\rangle|^2$

$$= \frac{1}{|v_A|} \cdot \frac{L^3}{2E_A} \cdot \frac{L^3}{2E_B} \cdot \frac{2\pi}{\hbar} \rho_f(E_i) |\langle f|V|i\rangle|^2$$

Recall that ρ_f contains L^{3n_f} when there are n_f final state particles.

$|\langle f|V|i\rangle|^2$ will introduce $\left(\frac{1}{L^3}\right)^{n_i}\left(\frac{1}{L^3}\right)^{n_f}$

where $n_i = 2$. So all factors of L cancel.

So, we can drop L from these expressions and write

$$\phi(x) = e^{i\vec{k}\cdot\vec{x}}$$
$$\phi(x,t) = e^{i\vec{k}\cdot\vec{x} - iEt}$$

Partial width, $d\Gamma_f = \frac{2\pi}{2M} \underbrace{\prod_i \frac{d^3p_i}{(2\pi)^3 \cdot 2E_i}}_{\text{product over final state particles.}} |\langle f|V|i\rangle|^2$

Cross section, $d\sigma(i \to f) = \frac{2\pi}{|v_{AB}| \cdot 2E_A \cdot 2E_B} \underbrace{\prod_i \frac{d^3p_i}{(2\pi)^3 \cdot 2E_i}}_{\text{product over final state particles}} |\langle f|V|i\rangle|^2$

What if we don't know the exact form of V?
Can we say anything about $\langle f|V|i\rangle$?
We can if we consider symmetries ...

Suppose we assume that V is translation invariant. Then if $\mathcal{U}(\bar{x})$ is a translation operator, $\mathcal{U}^{\dagger}(\bar{x})\, V\, \mathcal{U}(\bar{x}) = V$

Since $\mathcal{U}(\bar{x}) = e^{-i\bar{P}\cdot\bar{x}}$ V and P commute:

$$[V, P] = VP - PV = 0.$$

Suppose $|i\rangle$ is an eigenstate of P with eigenvalue p. ie, $P|i\rangle = p|i\rangle$. But then $V|i\rangle$ is also an eigenstate of P:

$$PV|i\rangle = VP|i\rangle = p\,V|i\rangle$$

moreover, it has the same eigenvalue. So $V|i\rangle = c|i\rangle$ where c is some complex number.

Then, $\langle f|V|i\rangle = \delta(p_f - p_i)\,\langle f\|V\|i\rangle$

where $\langle f\|V\|i\rangle$ is a "reduced matrix element" that does not depend on p.

Another example. Suppose V is rotationally symmetric. Then $R_z^\dagger(\theta)\, V\, R_z(\theta) = V$

$R_z(\theta) = e^{-iJ_z\theta}$, V commutes with J_z.

If $|j, j_z\rangle$ is an eigenstate of the angular momentum operators, that is

$$J^2 |j, j_z\rangle = j(j+1)|j, j_z\rangle$$

$$J_z |j, j_z\rangle = j_z |j, j_z\rangle$$

then

$$J_z V|j, j_z\rangle = V J_z |j, j_z\rangle$$

$$= j_z V|j, j_z\rangle$$

So $V|j, j_z\rangle$ is also an eigenstate of J_z.

Repeat this argument for J^2.

Hence, $\langle j' j_z' | V | j\, j_z\rangle = \delta_{jj'}\, \delta_{j_z j_z'} \langle j \| V \| j\rangle$

where $\langle j\|V\|j\rangle$ is a function of the total angular momentum only and does not depend on j_z.

We will come back to this shortly.

Another example: Parity.

$$P^{\dagger} V(x) P = V(-x)$$

$$P|x\rangle = |-x\rangle$$

If $|\psi\rangle$ is an eigenstate of P, then

$$P|\psi\rangle = p\,|\psi\rangle \quad \text{where} \quad p = \pm 1.$$

Parity can be intrinsic, or associated with spatial wavefunctions: If $|\psi\rangle$ describes a state of two particles described by relative spherical coordinates r, θ, ϕ then

$$\langle \vec{x} | \psi \rangle = R(r)\, Y_\ell^m(\theta, \phi)$$

$$\langle \vec{x} | P | \psi \rangle = R(r)\, Y_\ell^m(\pi - \theta, \phi + \pi) = (-1)^\ell\, Y_\ell^m(\theta, \phi)$$

Including the intrinsic parity, if the two particles have parity P_1 and P_2 then the total parity of the state is

$$P_1 P_2 (-1)^\ell$$

$\Rightarrow$ If the Hamiltonian is invariant under some symmetry transformation, then the eigenvalues of the generators of the transformation are good quantum numbers, conserved in interactions mediated by H.

Translation invariance : momentum
Rotation invariance : angular momentum (spin)
Space inversion : parity

are there others? If we find some, it tells us about H, and in particular, V.

The Strong Interaction.

- responsible for nuclear interactions
- responsible for interactions and properties of the <u>hadrons</u>.

Some hadrons are quasi-stable:

eg, neutron $\tau = 15$ min
kaons $\tau = 1.24 \times 10^{-8}$ sec

Others are not:

$$\Delta^+ \to n\pi^+, \; p\pi^0$$

$$\Gamma \sim 120 \text{ MeV}$$

$\tau = \frac{1}{\Gamma}$ but we need to insert factors of $\hbar$ to get this in seconds:

$$\tau = \frac{\hbar}{\Gamma} = \frac{6.58 \times 10^{-22} \text{ MeV} \cdot \text{s}}{120 \text{ MeV}} = 5.48 \times 10^{-24} \text{ sec}$$

too short to think of as a stable particle. These are called Hadronic <u>resonances</u>.

A large number of hadrons and hadronic resonances have been observed:

Baryons: spin 1/2

Σ^-, Σ^0, Σ^+, Λ^0

Ξ^-, Ξ^0

p, n

+ antiparticles.

spin 3/2

Δ^{++}, Δ^+, Δ^0, Δ^-

Σ^{*-}, Σ^{*0}, Σ^{*+}

Ξ^{*-}, Ξ^{*0}

Ω^-

Mesons: Spin 0 (pseudoscalar mesons)

K^0, K^+

π^-, π^0, π^+, η, η'

K^-, $\bar{K}^0$

Spin 1 (vector mesons)

ρ^-, ρ^0, ρ^+, ω, ϕ

K^{*0}, K^{*+}

$\bar{K}^{*0}$, K^{*-}

Can we make sense out of all of these states? We can classify them by mass and charge but what other quantum numbers do they have that result from symmetries of the strong interaction Hamiltonian?

Eg, can we explain why

$\sigma(\pi^+ + p)$ is three times larger than $\sigma(\pi^- + p)$?