Time evolution in quantum mechanics.

Suppose $|i\rangle$ is an eigenstate of a "free" Hamiltonian, H_0. That is, $H_0|i\rangle = E_i|i\rangle$.

The time evolution operator in the Schrödinger picture gives the state at a later time:

$$\begin{aligned}|i,t\rangle &= \mathcal{U}(t)|i\rangle \\ &= e^{-iH_0t/\hbar}|i\rangle \\ &= e^{-iE_it/\hbar}|i\rangle\end{aligned}$$

Of course, the probability of observing the system in state $|i\rangle$ at a later time is unity:

$$\begin{aligned}P &= |\langle i|i,t\rangle|^2 \\ &= |e^{-iE_it/\hbar}\langle i|i\rangle|^2 \\ &= 1\end{aligned}$$

This is a reasonable way to describe a stable particle that does not interact.

But an unstable particle should behave like $P(t) = e^{-\Gamma t}$ where $\Gamma = 1/\tau$.

So let's fix this up.

Suppose we write

$$\mathcal{U}(t) = e^{-iHt/\hbar - \Gamma t/2}$$

Then $|\langle i|i,t\rangle|^2$ = probability of observing the system in state $|i\rangle$ at time t.

$$|\langle i|i,t\rangle|^2 = |\langle i|e^{-iE_i t/\hbar - \Gamma t/2}|i\rangle|^2$$

$$= |e^{-iE_i t/\hbar - \Gamma t/2}|^2$$

$$= e^{-\Gamma t}$$

So that works, but only by definition.

What we would really like to calculate is the probability of observing the system in a different state, say $|f\rangle$, after we've waited long enough that the particle must have decayed.

The part of the Hamiltonian responsible for the decay, that is, the coupling of state $|i\rangle$ to other states $|f\rangle$, can be treated as a perturbation to the free Hamiltonian:

$$H = H_0 + V$$

We want to find an expression for the state at time t when it was initially in state $|i\rangle$ at time $t=0$.

In general, $|i,t\rangle$ is given in terms of the time evolution operator:

$$|i,t\rangle = \mathcal{U}(t)\,|i\rangle$$

We want to calculate the probability of observing a free particle state $|f\rangle$ at some time, t: $|f,t\rangle = e^{-iE_f t/\hbar}\,|f\rangle$,

So, we use these states to express $|i,t\rangle$:

$$\begin{aligned} |i,t\rangle &= \sum_n |n,t\rangle\langle n,t|i,t\rangle \\ &= \sum_n |n,t\rangle\langle n,t|\mathcal{U}(t)|i\rangle \\ &= \sum_n c_n(t)\,|n,t\rangle \end{aligned}$$

where $c_n(t) = \langle n,t|\mathcal{U}(t)|i\rangle$

[insert $P(f|i)$ at time t is just $|c_f(t)|^2$]

Recall that $\langle n,t| = e^{iE_n t/\hbar}\langle n|$
and that $\mathcal{U}(t) = e^{-iHt/\hbar}$ where $H = H_0 + V$.

Hence, $$\frac{dc_n(t)}{dt} = \frac{iE_n}{\hbar}\langle n,t|\mathcal{U}(t)|i\rangle - \frac{i}{\hbar}\langle n,t|(H_0+V)\mathcal{U}(t)|i\rangle$$

But $\langle n,t|H_0 = E_n\langle n,t|$

So $$\frac{dc_n(t)}{dt} = \frac{-i}{\hbar}\langle n,t|V\,\mathcal{U}(t)|i\rangle$$
$$= \frac{-i}{\hbar}\sum_m \langle n,t|V|m,t\rangle c_m(t)$$
$$= \frac{-i}{\hbar}\sum_m e^{i(E_n-E_m)t/\hbar}\langle n|V|m\rangle c_m(t).$$

When $t\to 0$ we expect that
$$c_i(t) \to e^{-iE_it/\hbar - \Gamma t/2}$$
$$c_j(t) \to 0 \quad \text{when } j\neq i.$$

So $$\frac{dc_n(t)}{dt} = \frac{-i}{\hbar}e^{i(E_n-E_i)t/\hbar}e^{-\Gamma t/2}\langle n|V|i\rangle$$

$$c_n(t) = \frac{-i}{\hbar}\int_0^t e^{i(E_n-E_i)t'/\hbar}e^{-\Gamma t'/2}\langle n|V|i\rangle$$

But if $V_{ni} = \langle n|V|i\rangle$ does not depend too strongly on energy, then we can take it out of the integral:

$$c_n(t) = \frac{-i}{\hbar} V_{ni} \int_0^t e^{i(E_n-E_i)t'/\hbar} e^{-\Gamma t'/2}$$

$$= \frac{-i}{\hbar} \frac{V_{ni}}{i(E_n-E_i)/\hbar - \Gamma/2} e^{i(E_n-E_i)t'/\hbar - \Gamma t'/2} \Big|_0^t$$

what we care about is the asymptotic behavior when $t \to \infty$.

In this case, $c_n(\infty) = \frac{i}{\hbar} \frac{V_{ni}}{i(E_n-E_i)/\hbar - \Gamma/2}$

The probability of observing the final state f at some time in the future is

$$|c_f(\infty)|^2 = \frac{|V_{fi}(E)|^2}{(E_f-E_i)^2 + (\Gamma\hbar/2)^2}$$

In practice we are usually interested in the probability of observing a particular class of final states. So we sum (integrate) over all kinematic configurations of final state, f:

$$P(i \to f) = \sum_f |c_f(\infty)|^2 = \int_0^\infty \frac{dE\, \rho_f(E)\, |V_{fi}(E)|^2}{(E-E_i)^2 + (\Gamma\hbar/2)^2}$$

where $\rho_f(E)$ is the number of final states with energy E.

Since the integrand is very small except in the region where $E \approx E_i$, this is

$$P(i \to f) = \rho_f(E_i)\,|V_{fi}(E_i)|^2 \int_{-\infty}^{\infty} \frac{dE}{(E-E_i)^2 + (\Gamma\hbar/2)^2}$$

$$= \rho_f(E_i)\,|V_{fi}(E_i)|^2 \cdot \frac{2\pi}{\Gamma\hbar}$$

If we sum over ALL final states then

$$\sum_f P(i \to f) = 1$$

We call the probability $P(i \to f)$ the branching fraction, and the partial width,

$$\Gamma_f = \Gamma\, BF(i \to f)$$

So that $$\sum_f \Gamma_f = \Gamma = \sum_f \rho_f(E_i)\,|V_{fi}(E_i)|^2 \cdot \frac{2\pi}{\hbar}$$

This is called Fermi's Golden Rule even though it originated with Dirac.

Two things remain:

1. What exactly is $\rho_f(E)$?
2. What exactly is $\langle f|V|i\rangle$?

Consider a decay to a specific final state:

$$\Gamma_f = \rho_f(E_i)\,|V_{fi}(E_i)|^2 \cdot \frac{2\pi}{\hbar}$$

where $V_{fi}(E_i) = \langle f \,|\, V \,|\, i \rangle$

V_{fi} and ρ_f must be defined in a consistent way:

Consider a particle confined to a box with sides of length L. If we write

$$\phi(\vec{x}) = \langle \vec{x} \,|\, \phi \rangle = N e^{i\vec{k}\cdot\vec{x}} \qquad \phi(\vec{x},t) = N e^{i\vec{k}\cdot\vec{x} - iEt}$$

where
$$k_x = 2\pi n_x / L$$
$$k_y = 2\pi n_y / L$$
$$k_z = 2\pi n_z / L$$

$$\Delta n_x = \frac{\Delta k_x L}{2\pi}, \quad \Delta n_y = \frac{\Delta k_y L}{2\pi}, \quad \Delta n_z = \frac{\Delta k_z L}{2\pi}$$

$$\Delta n_x \Delta n_y \Delta n_z = \frac{\Delta k_x \Delta k_y \Delta k_z}{(2\pi)^3} L^3$$

Probability density is $\rho = i\left(\phi^*(x)\frac{\partial \phi}{\partial t} - \phi(x)\frac{\partial \phi^*}{\partial t}\right)$

$$= 2E\,\phi^*(x)\,\phi(x) = 2E|N|^2$$

If we integrate this over the whole box:

$$= \int_V d^3x\,\rho(x) = 2E|N|^2 L^3 =$$

$$\Rightarrow N = \frac{1}{\sqrt{V}}$$

final states per volume is $\frac{d^3k}{(2\pi)^3} L^3$

particles per volume is $2E$

final states per particle is $\frac{d^3k}{(2\pi)^3} \cdot \frac{L^3}{2E}$

In general, $\rho_f = \frac{d^3p_1 L^3}{(2\pi)^3 \cdot 2E_1} \frac{d^3p_2 L^3}{(2\pi)^3 \cdot 2E_2} \cdots \delta(E_f - E_i)$

Factors of L^3 will cancel because of the normalization of the states in $\langle f|V|i\rangle$.

Recall $\phi(x) = \frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{V}} = \frac{e^{i\vec{k}\cdot\vec{x}}}{L^{3/2}}$

Each final state particle introduces L^{-3} when we evaluate $|\langle f|V|i\rangle|^2$.

What about L^{-3} from the initial state?

Suppose a particle is decaying. Then as written Γ is a decay rate per unit volume. If we integrate the initial state over the volume V, then we get $2EL^3$. But we assume there was only one particle.

Bottom line: All vactors of L^3 cancel and the result does not depend on the volume of the box.