25 Oct 2005

$$u^{(s)}(\vec{p}) = \begin{pmatrix} \sqrt{E+m}\;\chi^{(s)} \\ \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\;\chi^{(s)} \end{pmatrix} \qquad E = \sqrt{|\vec{p}|^2 + m^2}$$

$$v^{(s)}(\vec{p}) = \begin{pmatrix} \dfrac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\;\chi^{(s)} \\ \sqrt{E+m}\;\chi^{(s)} \end{pmatrix} \qquad E = \sqrt{|\vec{p}|^2 + m^2}$$

Adjoint solutions are $\bar{u}^{(s)}(\vec{p}) = u^{\dagger(s)}(\vec{p})\,\gamma^0$

$$\bar{u}^{(s)}(\vec{p}) = \left(\sqrt{E+m}\;\chi^{(s)T} ,\ \frac{-\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\;\chi^{(s)T} \right)$$

$$\bar{v}^{(s)}(\vec{p}) = \left(\frac{\vec{\sigma}\cdot\vec{p}}{\sqrt{E+m}}\;\chi^{(s)T} ,\ -\sqrt{E+m}\;\chi^{(s)T} \right)$$

Very useful results:

$$\sum_{s=1,2} u^{(s)}(\vec{p})\bar{u}^{(s)}(\vec{p}) = \sum_{s=1,2} \begin{pmatrix} (E+m)\,\chi^{(s)}\chi^{(s)T} & -\vec{\sigma}\cdot\vec{p}\,\chi^{(s)}\chi^{(s)T} \\ \vec{\sigma}\cdot\vec{p}\,\chi^{(s)}\chi^{(s)T} & \dfrac{-|\vec{p}|^2}{E+m}\,\chi^{(s)}\chi^{(s)T} \end{pmatrix}$$

$$= \begin{pmatrix} E+m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E+m \end{pmatrix}$$

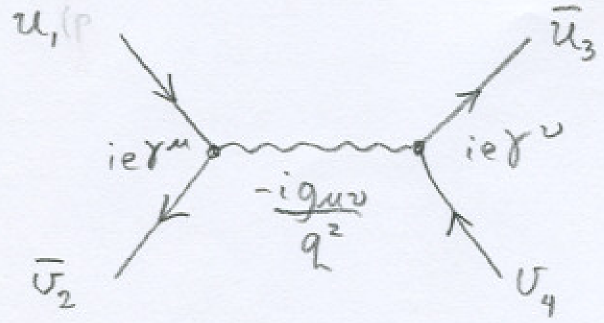
$$= p^{\mu}\gamma_{\mu} + m$$

$$= \not{p} + m$$

Similarly, $\displaystyle\sum_{s=1,2} v^{(s)}(\vec{p})\bar{v}^{(s)}(\vec{p}) = \not{p} - m$.

These relationships allow us to simplify the invariant amplitude.

Calculation of $e^+e^- \to \mu^+\mu^-$.



Rules: Incomming electron $u^{(s)}(p)$

Incomming positron $\bar{v}^{(s)}(p)$

Outgoing electron $\bar{u}^{(s)}(p)$

Outgoing positron $v^{(s)}(p)$

Each vertex $ie\gamma^\mu$

Photon propagator: $\frac{-ig_{\mu\nu}}{q^2}$

$$-i\mathcal{M} = \left(ie\bar{v}_2\gamma^\mu u_1\right)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\left(ie\bar{u}_3\gamma^\nu v_4\right)$$

$$= \frac{ie^2}{q^2}\left(\bar{v}_2\gamma^\mu u_1\right)\left(\bar{u}_3\gamma_\mu v_4\right)$$

Where $u_1 \equiv u_1^{(s_1)}(p_1)$, etc.

But what we need to calculate is $|\mathcal{M}|^2$.

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left[(\bar{v}_2\gamma^\mu u_1)(\bar{u}_3\gamma_\mu v_4)\right]\left[(\bar{v}_2\gamma^\nu u_1)(\bar{u}_3\gamma_\nu v_4)\right]^*$$

But $*$ is the same as $\dagger$ here:

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left[(\bar{v}_2\gamma^\mu u_1)(\bar{u}_3\gamma_\mu v_4)\right]\left[(\bar{u}_1\gamma^\nu v_2)(\bar{v}_4\gamma_\nu u_3)\right]$$

since
$$\begin{aligned}
\left[\bar{u}\gamma^\mu v\right]^\dagger &= v^\dagger\gamma^{\mu\dagger}\bar{u}^\dagger \\
&= v^\dagger\gamma^0\gamma^\mu\gamma^0\gamma^{0\dagger}u \\
&= \bar{v}\gamma^\mu\gamma^0\gamma^0\gamma^0\gamma^0 u \\
&= \bar{v}\gamma^\mu u
\end{aligned}$$

If we sum over final state spins and average over initial state spins then we can write

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{q^4} L_e^{\mu\nu} L^{\text{muon}}_{\mu\nu}$$

where
$$\begin{aligned}
L_e^{\mu\nu} &= \frac{1}{4}\sum_{s_1,s_2}\left[\bar{v}_2\gamma^\mu u_1\bar{u}_1\gamma^\nu v_2\right] \\
&= \frac{1}{4}\sum_{s_1,s_2}\mathrm{Tr}\left[v_2\bar{v}_2\gamma^\mu u_1\bar{u}_1\gamma^\nu\right] \\
&= \frac{1}{4}\mathrm{Tr}\left[(\not{p}_2-m)\gamma^\mu(\not{p}_1+m)\gamma^\nu\right]
\end{aligned}$$

Similarly, $$L^{muon}_{\mu\nu} = \sum_{s_3 s_4} \left[\bar{u}_3 \gamma_\mu v_4 \bar{v}_4 \gamma_\nu u_3 \right]$$
$$= \sum_{s_3, s_4} Tr\left[u_3 \bar{u}_3 \gamma_\mu v_4 \bar{v}_4 \gamma_\nu \right]$$
$$= Tr\left[(\not{p}_3 + M) \gamma_\mu (\not{p}_4 - M) \gamma_\nu \right]$$

Traces of products of γ-matrices can be easily derived or looked up in tables. See section 6.4 of H&M.

$$L_e^{\mu\nu} = \tfrac{1}{4} Tr\left[\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu \right] + \tfrac{1}{4} m\, Tr\left[\not{p}_2 \gamma^\mu \gamma^\nu \right]$$
$$- \tfrac{1}{4} m\, Tr\left[\gamma^\mu \not{p}_1 \gamma^\nu \right]$$
$$- \tfrac{1}{4} m^2\, Tr\left[\gamma^\mu \gamma^\nu \right]$$

Use the results $Tr(\gamma^\mu \gamma^\nu) = 4 g^{\mu\nu}$

$Tr(\text{odd \# of } \gamma\text{-matrices}) = 0$

$Tr(\not{p}_2 \gamma^\mu \not{p}_1 \gamma^\nu) = 4\left[p_2^\mu p_1^\nu - p_1 \cdot p_2\, g^{\mu\nu} + p_1^\mu p_2^\nu \right]$

So $$L_e^{\mu\nu} = p_2^\mu p_1^\nu + p_1^\mu p_2^\nu - (p_1 \cdot p_2 + m^2) g^{\mu\nu}$$

Similarly, $$L^{muon}_{\mu\nu} = 4\left[p_{4\mu} p_{3\nu} + p_{3\mu} p_{4\nu} - (p_3 \cdot p_4 + M^2) g_{\mu\nu} \right]$$

Finally,

$$|\overline{\mathcal{M}}|^2 = \frac{4e^4}{q^4}\Big[2(p_2\cdot p_4)(p_1\cdot p_3) + 2(p_2\cdot p_3)(p_1\cdot p_4) - 2(p_1\cdot p_2)\big((p_3\cdot p_4)+M^2\big)$$
$$- 2(p_3\cdot p_4)\big((p_1\cdot p_2)+m^2\big) + 4\big((p_1\cdot p_2)+m^2\big)\big((p_3\cdot p_4)+M^2\big)\Big]$$

Suppose we ignore the masses. Then

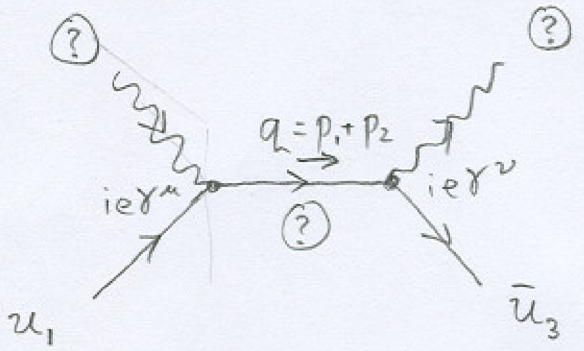
$$|\overline{\mathcal{M}}|^2 = \frac{8e^4}{q^4}\Big[(p_1\cdot p_3)(p_2\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3)\Big]$$

Write this using Mandlestam variables:

$$s = (p_1+p_2)^2 \approx 2p_1\cdot p_2 \approx 2p_3\cdot p_4$$
$$t = (p_1-p_3)^2 \approx -2p_1\cdot p_3 \approx -2p_2\cdot p_4$$
$$u = (p_1-p_4)^2 \approx -2p_1\cdot p_4 \approx -2p_2\cdot p_3$$
$$q^2 = (p_1+p_2)^2 = s$$

So $|\overline{\mathcal{M}}|^2 = \dfrac{2e^4}{s^2}\left(t^2+u^2\right)$ (H&M, eqn 6.31)

Other processes, eg, Compton scattering or e^+e^- annihilation:



More rules:

Incomming photon : $\varepsilon_\mu^{(\lambda)}(k)$

Outgoing photon : $\varepsilon_\mu^{*(\lambda)}(k)$

where $\varepsilon_\mu^{(+1)} = -\sqrt{\frac{1}{2}}(0, 1, i, 0)$ right

$\varepsilon_\mu^{(-1)} = \sqrt{\frac{1}{2}}(0, 1, -i, 0)$ left

circular polarization basis.

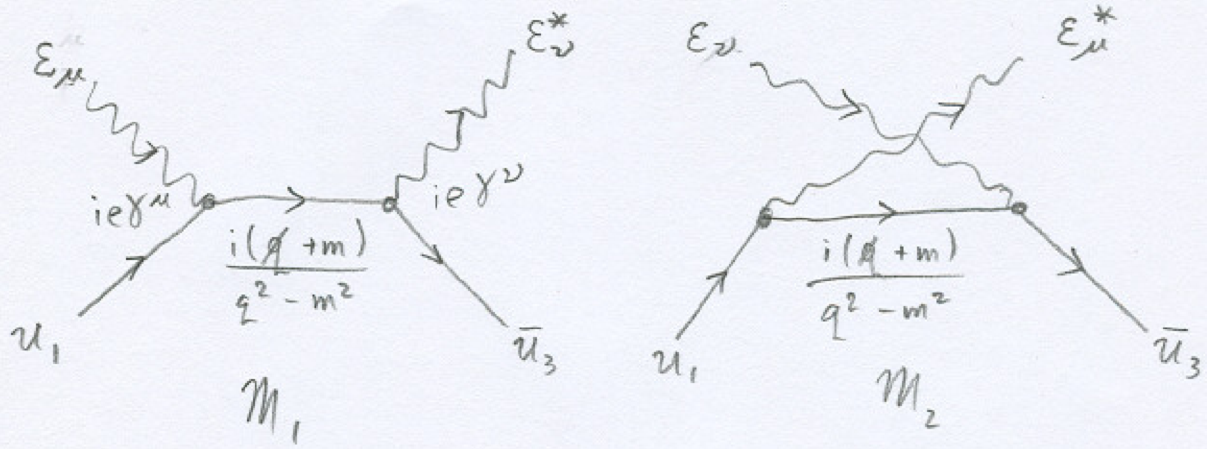
Electron propagator :

$$\frac{i(\not{q} + m)}{q^2 - m^2}$$

Notice that $\sum_\lambda \varepsilon_\mu^{*(\lambda)} \varepsilon_\nu^{(\lambda)} \rightarrow -g_{\mu\nu}$

simplifies calculation of $|\overline{\mathcal{M}}|^2$.

Two diagrams for Compton scattering



$$-i\mathcal{M}_1 = \bar{u}_3(ie\gamma^\nu)\left(\frac{i(\not{q}+m)}{q^2-m^2}\right)(ie\gamma^\mu)\,u_1\,\varepsilon_\mu\,\varepsilon^*_\nu$$

$$= \frac{-ie^2}{q^2-m^2}\left(\bar{u}_3\gamma^\nu(\not{q}+m)\gamma^\mu u_1\right)\left(\varepsilon_\mu\,\varepsilon^*_\nu\right)$$

$$|\overline{\mathcal{M}}_1|^2 = \frac{\frac{1}{4}e^4}{(q^2-m^2)^2}\sum_{\substack{s,s'\\ \lambda\lambda'}}\left(\bar{u}_3^{(s)}\gamma^\nu(\not{q}+m)\gamma^\mu u_1^{(s')}\right)\left(\bar{u}_1^{(s')}\gamma^{\mu'}(\not{q}+m)\gamma^{\nu'}u_3^{(s)}\right)$$
$$\times\left(\varepsilon_\mu^{(\lambda)}\,\varepsilon_\nu^{*(\lambda')}\right)\left(\varepsilon_{\mu'}^{*(\lambda)}\,\varepsilon_{\nu'}^{(\lambda')}\right)$$

$$= \frac{\frac{1}{4}e^4}{(q^2-m^2)^2}\,\mathrm{Tr}\left[(\not{p}_3+m)\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_1+m)\gamma^{\mu'}(\not{q}+m)\gamma^{\nu'}\right]$$
$$\times(-g_{\mu\mu'})(-g_{\nu\nu'})$$

$$= \frac{\frac{1}{4}e^4}{(q^2-m^2)^2}\,\mathrm{Tr}\left[(\not{p}_3+m)\gamma^\nu(\not{q}+m)\gamma^\mu(\not{p}_1+m)\gamma_\mu(\not{q}+m)\gamma_\nu\right]$$

If we ignore the electron mass, then

$$|\overline{\mathcal{M}}_1|^2 = \frac{\frac{1}{4}e^4}{q^4} \mathrm{Tr}\left[\not{p}_3 \gamma^\nu \not{q} \gamma^\mu \not{p}_1 \gamma_\mu \not{q} \gamma_\nu \right]$$

$$= \frac{\frac{1}{4}e^4}{q^4} \mathrm{Tr}\left[\gamma_\nu \not{p}_3 \gamma^\nu \not{q} \gamma^\mu \not{p}_1 \gamma_\mu \not{q} \right]$$

Another identity:

$$\gamma^\mu \not{p} \gamma_\mu = -2\not{p}$$

So $|\overline{\mathcal{M}}_1|^2 = \dfrac{4e^4}{q^4} \mathrm{Tr}\left[\not{p}_3 \not{q} \not{p}_1 \not{q} \right]$

where $q = p_1 + k_1$

$$|\overline{\mathcal{M}}_1|^2 = \frac{e^4}{q^4} \mathrm{Tr}\left[\not{p}_3 (\not{p}_1 + \not{k}_1) \not{p}_1 (\not{p}_1 + \not{k}_1) \right]$$

$$= \frac{e^4}{q^4} \Big[\cancelto{0}{\mathrm{Tr}\left[\not{p}_3 \not{p}_1 \not{p}_1 \not{p}_1 \right]} + \mathrm{Tr}\left[\not{p}_3 \not{p}_1 \not{p}_1 \not{k}_1 \right] + \mathrm{Tr}\left[\not{p}_3 \not{k}_1 \not{p}_1 \not{p}_1 \right] + \mathrm{Tr}\left[\not{p}_3 \not{k}_1 \not{p}_1 \not{k}_1 \right] \Big]$$

$$= \frac{e^4}{q^4} \mathrm{Tr}\left[\not{p}_3 \not{k}_1 \not{p}_1 \not{k}_1 \right]$$

$$= \frac{4e^4}{q^4} \left[(p_3 \cdot k_1)(p_1 \cdot k_1) - (p_1 \cdot p_3)(k_1^2) + (p_3 \cdot k_1) \cdot (p_1 \cdot k_1) \right]$$

$$= \frac{8e^4}{q^4} (p_1 \cdot k_1)(p_3 \cdot k_1)$$

Mandlestam variables:

$$s = (p_1 + k_1)^2 \approx 2 p_1 \cdot k_1$$

$$t = (p_1 - p_3)^2$$

$$u = (p_3 - k_1)^2 \approx -2 k_1 \cdot p_1$$

So $$\overline{|\mathcal{M}_1|^2} = -\frac{2e^4}{s}\left(us\right)$$

$$= -2e^4\left(\frac{u}{s}\right)$$

Similarly, $$\overline{|\mathcal{M}_2|} = -2e^4\left(\frac{s}{u}\right)$$

$$\overline{\mathcal{M}_1^* \mathcal{M}_2} = 0$$

So $$\overline{|\mathcal{M}|^2} = -2e^4\left(\frac{u}{s} + \frac{s}{u}\right)$$