6 - Oct - 2005

We need to develop a formalism for making quantitative calculations of matrix elements. We calculated the transition rate

$$\Gamma = \frac{2\pi}{\hbar} \rho_f(E_i) |\langle f | V | i \rangle|^2$$

under the assumption that V was a "small" perturbation to the free particle Hamiltonian. For electromagnetic interactions we will find that V can be expanded as a power series in $\alpha = \frac{e^2}{4\pi\hbar c} \approx \frac{1}{137}$.

So the second order terms are indeed quite small ($\alpha^2 < 10^{-4}$). For the strong interaction, $\alpha_s \sim 0.2$ so $\alpha_s^2 \sim .04$ which is maybe not so small that we can trust this expansion.

So we will concentrate for now on electromagnetic interactions and generalize the results to the weak and strong interactions later.

First, we need to describe free particle states. Their wavefunctions should satisfy Schrödinger's equation

$$H|\phi\rangle = E|\phi\rangle$$

In the position representation,

$$\langle\vec{x}|H|\phi\rangle = E\langle\vec{x}|\phi\rangle$$

$$H\,\phi(\vec{x},t) = E\,\phi(\vec{x},t)$$

The Hamiltonian should be consistent with special relativity:

$$H = \sqrt{p^2 + m^2} \quad \text{where } p \to -i\nabla$$

and $E \to i\frac{\partial}{\partial t}$. The square root is hard to work with so we consider

$$H^2\phi(x) = E^2\phi(x)$$

$$\text{or } (-\nabla^2 + m)\phi(\vec{x},t) = -\frac{\partial^2\phi(\vec{x},t)}{\partial t^2}$$

$$\text{or } \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi(\vec{x},t) = 0$$

which can be written using 4-vectors:

$$(\partial^\mu\partial_\mu + m^2)\phi(\vec{x},t) = 0$$

recall that $\partial^\mu = \left(\frac{\partial}{\partial t}, -\nabla\right)$

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A solution to this equation is

$$\phi(\vec{x},t) = N e^{-ip^\mu x_\mu}$$

where $p = (E, \vec{p})$

$x = (t, \vec{x})$

So $\partial^\mu \phi(x) = -ip^\mu \phi(x)$

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Check this explicitly:

$$\partial^\mu \phi(x) = \left(\frac{\partial}{\partial t}, -\nabla\right) \cdot N e^{-i(Et - \vec{p}\cdot\vec{x})}$$

$$= \left(-iE, -i\vec{p}\right) \phi(x)$$

$$= -ip^\mu \phi(x)$$

$$\partial_\mu \phi(x) = \left(\frac{\partial}{\partial t}, \nabla\right) \cdot N e^{-i(Et - \vec{p}\cdot\vec{x})}$$

$$= -i\left(E, -\vec{p}\right) \cdot N e^{-i(Et - \vec{p}\cdot\vec{x})}$$

$$= -ip_\mu \phi(x).$$

So $\partial^\mu \partial_\mu \phi(x) = -p^\mu p_\mu \phi(x)$

$= -m^2 \phi(x)$

So $\phi(x)$ is indeed a solution:

$$(\partial^\mu \partial_\mu + m^2)\,\phi(x) = 0.$$

However, there is no reason why E must be greater than zero, so can this solution really describe a physical state if it allows negative energies?

Furthermore, we can construct the probability density current: (by writing)

$$-i\phi^* (\partial^\mu \partial_\mu + m^2)\phi + i\phi(\partial^\mu \partial_\mu + m^2)\phi^* = 0$$

$$-i\phi^* \partial^\mu \partial_\mu \phi + i\phi \partial^\mu \partial_\mu \phi^* = 0$$ (the m^2 terms cancel)

$$\partial_\mu [i\phi^* \partial^\mu \phi - i\phi \partial^\mu \phi^*] = 0$$

or $\partial_\mu j^\mu(x) = 0$. $j^\mu(x)$ is a conserved current.

$$\frac{\partial}{\partial t} j^0(x) + \nabla \vec{j}(x) = 0$$

$$j^0 = i\phi^* \frac{\partial \phi}{\partial t} - i\phi \frac{\partial \phi^*}{\partial t}$$

represents the probability density (Why?) and $\vec{j}$ represents the flux of probability density.

But if $\phi(x) = Ne^{-ip^\mu x_\mu}$ then

$$j^0 = |N|^2 \cdot 2E$$

and $\vec{j} = |N|^2 \cdot 2\vec{p}$.

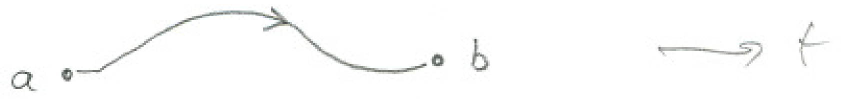
If $E < 0$ then is j^0 really probability density?

There are two ways out:

1. Interpret $E < 0$ solutions as antiparticles. Then $-ej^{\mu}(x)$ represents charge current density and everything is fine.

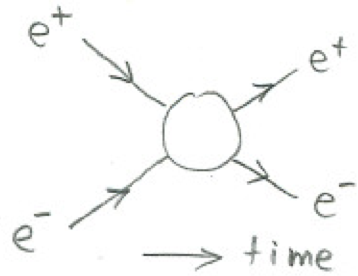
2. A negative energy (anti?) particle propagating (backward) forward in time is equivalent to a positive energy anti-particle propagating (forward) backwards in time.

   Example: a negative charge moving from ⓐ to ⓑ

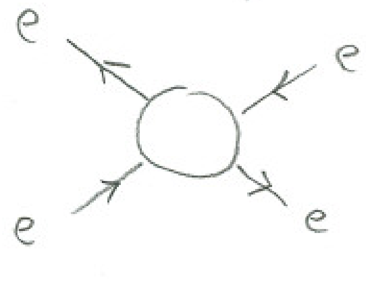


   is the same as a positive charge moving from b to a.

So when we want to calculate processes involving antiparticles:



we actually calculate this:



where all lines are electrons and the arrows indicate the flow of electric charge.

So now we think we can describe free particles. What about interactions? For electromagnetic interactions we replace p^μ by $p^\mu + eA^\mu$ where A^μ is the electromagnetic 4-potential.

Then, $$p^2 = p^\mu p_\mu \to (p^\mu + eA^\mu)(p_\mu + eA_\mu)$$
$$= p^\mu p_\mu + e p^\mu A_\mu + e A^\mu p_\mu + e^2 A^\mu A_\mu .$$
$$= -\partial^\mu \partial_\mu - ie\partial^\mu A_\mu - ieA^\mu \partial_\mu + e^2 A^\mu A_\mu$$

So $$(p^2 - m^2)\phi(x) = (-\partial^\mu\partial_\mu - m^2)\phi(x) - ie\partial^\mu(A_\mu \phi(x)) - ieA^\mu \partial_\mu \phi(x) + e^2 A^\mu A_\mu \phi(x) .$$

Since we will expand in a power series in $\frac{e^2}{4\pi}$ we drop the last term, since it will be small compared with the $-ieA$ terms.

So a charged particle moving in an electromagnetic field satisfies the differential equation

$$(\partial^\mu \partial_\mu + m^2)\,\phi(x) = -ie(\partial^\mu A_\mu + A^\mu \partial_\mu)\,\phi(x)\,.$$

$$= -V\,\phi(x)$$

where $V = -ie(\partial^\mu A_\mu + A^\mu \partial_\mu)$.