Lecture 4

4-component vectors like

$$p = (E, \vec{p}c)$$

and

$$x = (ct, \vec{x})$$

transform under Lorentz transformations (boosts) along the x-axis as follows:

$$p' = \begin{pmatrix} E' \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta & & 0 \\ \gamma\beta & \gamma & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

In general, we write the first component as x^0 and $\vec{x}$ as x^i $\to i = 1, 2, 3$. Collectively we label all four components as x^μ, $\mu = 0, 1, 2, 3$.

We can write the elements of the transformation matrix as

$$L^\mu{}_\nu$$

Einstein's repeated index convention: A repeated index has an implicit summation.

$$x'^\mu = L^\mu{}_\nu x^\nu \equiv \sum_{\nu=0}^{3} L^\mu{}_\nu x^\nu$$

Summation is performed only between upper and lower indices. There is a relative minus sign between their space components.

$$x_\mu = (x^0, -x^1, -x^2, -x^3)$$

$$x_\mu = g_{\mu\nu} x^\nu \qquad g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix}$$

(Metric tensor)

Lorentz invariant quantities:

$$x^\mu x_\mu = g_{\mu\nu} x^\mu x^\nu$$

is a Lorentz invariant — the same in all reference frames.

Example:

$$p^\mu p_\mu = E^2 - \vec{p}\cdot\vec{p}\,c^2$$

$$= E^2 - |\vec{p}|^2 c^2 = m^2 c^4 .$$

m is the invariant mass.

$$x^\mu x_\mu = c^2 t^2 - |\vec{x}|^2$$

This is zero for any particle moving at the speed of light.

Other conventions exist ... beware!

$$-s^2 = \vec{x}\cdot\vec{x} - (x^0)^2 \qquad g^{\mu\nu} = \begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$$

Another convention:

$$x = (x^1, x^2, x^3, x^4)$$

$$x^4 = i x^0$$

Then $x \cdot x = |\vec{x}|^2 - (x^0)^2 \qquad g = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & 1 \end{pmatrix}$

We will only use the $+,-,-,-$ convention.

In quantum mechanics, E and $\vec{p}$ are associated with operators:

$$E \to i\hbar \frac{\partial}{\partial t}$$

$$\vec{p} \to -i\hbar \vec{\nabla}$$

So we can write $p^\mu \to i\hbar \partial^\mu$

where $\partial^\mu = \left(\frac{\partial}{\partial t}, -\vec{\nabla} \right)$

(careful...)

$$\partial_\mu = \left(\frac{\partial}{\partial t}, +\vec{\nabla} \right)$$

Note that the dimensions are inconsistent as written. We should probably put in a factor of c, but it is easier to leave them out and absorb them into the units. Likewise with $\hbar$.

This is like defining a system of units in which $\hbar$ and c have unit dimensions.

In this notation we just write

$$s^2 = t^2 - |\vec{x}|^2$$

$$m^2 = E^2 - |\vec{p}|^2$$

and remember that m has dimensions of MeV$/c^2$ and $\vec{p}$ has dimensions MeV$/c$.

Particle decays.

$$A \to a + b + c + \cdots$$

Momentum is always conserved.

$$p_A^\mu = p_a^\mu + p_b^\mu + p_c^\mu + \cdots$$

or $E_A = E_a + E_b + \cdots$
$\vec{p}_A = \vec{p}_a + \vec{p}_b + \cdots$

A common situation: We know m_A, m_a, m_b and we want to know the final state energies when A decays at rest.

$$p_A = p_a + p_b$$

$$M_A^2 = p_A^2 = (p_a + p_b)^2$$
$$= p_a^2 + 2 p_a \cdot p_b + p_b^2$$
$$= m_a^2 + 2 p_a \cdot p_b + m_b^2$$

But, $p_b = p_A - p_a$

$$p_a \cdot p_b = p_a \cdot (p_A - p_a) = p_a \cdot p_A - m_a^2$$

If A decays at rest then $p_A = (M_A, \vec{0})$

So $p_a \cdot p_A = M_A E_a$.

$$M_A^2 = m_a^2 + 2(M_A E_a - m_a^2) + m_b^2$$
$$= 2 M_A E_a + m_b^2 - m_a^2$$

$$E_a = \frac{M_A^2 + m_a^2 - m_b^2}{2M_A}$$

Likewise

$$E_b = \frac{M_A^2 + m_b^2 - m_a^2}{2M_A}.$$

The momentum can be calculated using

$$|\vec{p}| = \sqrt{E^2 - m^2}$$

$$|\vec{p}_a| = |\vec{p}_b| = (E_a^2 - m_a^2)^{1/2}$$

$$= (\text{algebra})$$

$$= \frac{[(M_A^2 - (m_a + m_b)^2)(M_A^2 - (m_a - m_b)^2)]^{1/2}}{2M_A}.$$

What if the decaying particle is not at rest? Calculate the Lorentz boost parameters

$$\gamma = E_A / M_A$$

$$\beta = |\vec{p}_A| / E_A$$

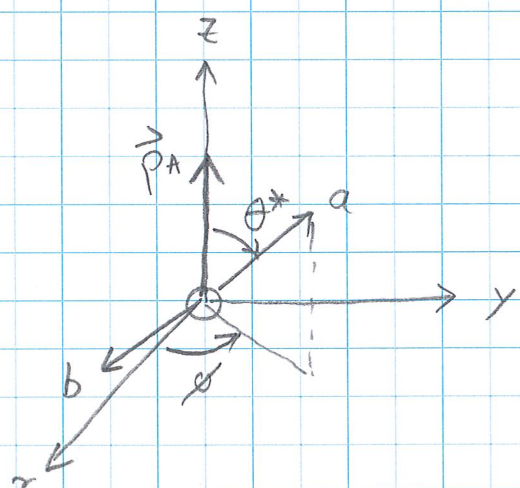
$$\gamma\beta = |\vec{p}_A| / M_A$$

Calculate $|\vec{p}_a^{\,*}|$ and $|\vec{p}_b^{\,*}|$ in the rest frame of A. If we use a coordinate system where the z-axis is along $\vec{p}_A$ then we can write

$$p^*_{ax} = |\vec{p}^{\,*}| \sin\theta^* \cos\phi$$

$$p^*_{ay} = |\vec{p}^{\,*}| \sin\theta^* \sin\phi$$

$$p^*_{az} = |\vec{p}^{\,*}| \cos\theta^*$$



p^*_{ax} and p^*_{ay} are not affected by a Lorentz boost along the z-axis.

In the lab frame,

$$E_a = \gamma E_a^* + \gamma\beta p_{az}^* = \gamma E_a^* + \gamma\beta |\vec{p}^*| \cos\theta^*$$

$$p_{az} = \gamma\beta E_a^* + \gamma p_{az}^* = \gamma\beta E_a^* + \gamma |\vec{p}^*| \cos\theta^*$$

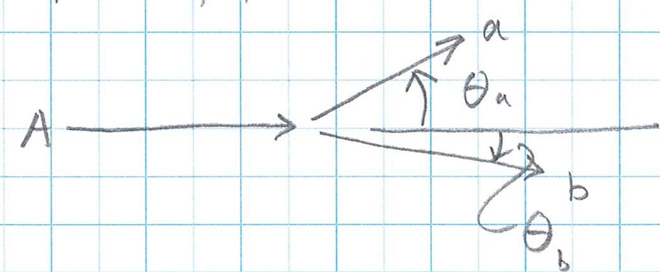
There might be messy expressions, but you can calculate them this way.

Computers are good at this type of thing.

The transverse components are often denoted

$$p_T = \sqrt{p_{ax}^2 + p_{ay}^2} = \sqrt{p_{bx}^2 + p_{by}^2}$$

then, $\tan\theta_a = \dfrac{p_T}{p_{az}}$



For particle b, $p_{bx}^* = -p_{ax}^*$

$$p_{by}^* = -p_{by}^*$$

$$p_{bz}^* = -p_{bz}^* = -|\vec{p}^*| \cos\theta^*$$

$$E_b = \gamma E_b^* - \gamma\beta |\vec{p}^*| \cos\theta^*$$

$$p_{bz} = \gamma\beta E_b^* - \gamma |\vec{p}^*| \cos\theta^*$$

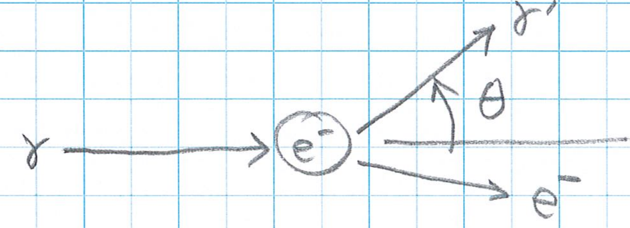
$$\tan\theta_b = \frac{p_T}{p_{bz}}$$

Again, these aren't necessarily nice expressions, but this is how you calculate them.

Collisions $\quad A + B \to a + b + \cdots$

Example, Compton scattering.

A photon scatters from a stationary electron. What is the relation between the energy and angle of the scattered photon?



$p_\gamma = (E, 0, 0, E) \qquad |\vec{p}| = E$ since $m=0$

$p_e = (m_e, 0, 0, 0)$ at rest

$p'_\gamma = (E', E'\sin\theta, 0, E'\cos\theta)$

$p'_e = p_\gamma + p_e - p'_\gamma$

$p'_e = (E + m_e - E', -E'\sin\theta, 0, E - E'\cos\theta)$

$$|\vec{p}_{e'}|^2 = (\vec{p}_\gamma - \vec{p}_{\gamma'})\cdot(\vec{p}_\gamma - \vec{p}_{\gamma'}) = E_{e'}^2 - m_e^2$$

$$= \cancel{E^2} + E'^2 - 2EE'\cos\theta$$

$$= E_{e'}^2 - m_e^2$$

$$= (E + m_e - E')^2 - m_e^2$$

$$= \cancel{E^2} + \cancel{E'^2} - 2EE' + 2Em_e - 2E'm_e + \cancel{m_e^2} - \cancel{m_e^2}$$

$$-2EE'\cos\theta = -2EE' + 2(E - E')m_e$$

$$(E - E')m_e = EE'(1 - \cos\theta)$$

$$1 - \cos\theta = \left(\frac{1}{E'} - \frac{1}{E}\right) m_e$$

$$\frac{1}{E'} - \frac{1}{E} = \frac{1-\cos\theta}{m_e}$$

Recall that $E = \frac{hc}{\lambda}$ and we replace m_e by $m_e c^2$ to get the units right.

Then $\frac{\lambda'}{hc} - \frac{\lambda}{hc} = \frac{1-\cos\theta}{m_e c^2}$

$$\lambda' - \lambda = \frac{h}{m_e c}(1 - \cos\theta)$$