Physics 536 - Summary of ODE's for RLC circuits.

The RLC circuit discussed in class led to the integral-differential equation

$$L\frac{di}{dt} + Ri(t) + \frac{1}{C}\left(Q_0 + \int_0^t i(t)dt\right) = V \tag{1}$$

but by differentiating once we obtained

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i(t) = 0 \tag{2}$$

which is a *linear, second-order, homogeneous* differential equation with *constant coefficients*. One can hypothesize that a solution to this differential equation might be of the form $i(t) = e^{\alpha t}$ which can be substituted into the equation to discover what conditions the constant α must satisfy. Thus,

$$\frac{di}{dt} = \alpha i(t) \tag{3}$$

$$\frac{d^2i}{dt^2} = \alpha^2 i(t) \tag{4}$$

and therefore, $(L\alpha^2 + R\alpha + 1/C)i(t) = 0$. Since $i(t) \neq 0$, α must be a root of the polynomial

$$L\alpha^2 + R\alpha + 1/C = 0. \tag{5}$$

These roots are

$$\alpha_+ = \frac{-R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \tag{6}$$

$$\alpha_- = \frac{-R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}. \tag{7}$$

and a general solution to the differential equation is

$$i(t) = Ae^{\alpha_+ t} + Be^{\alpha_- t} \tag{8}$$

where A and B are constants of integration.

The algebraic sign of the expression $R^2/4L^2 - 1/LC$ determins whether the roots are purely real or whether they are complex numbers.

If $R^2/4L^2 \ll 1/LC$, then $\alpha_\pm \approx -\gamma \pm i\omega_0$ where

$$\gamma = \frac{R}{2L} \tag{9}$$

$$\omega_0 = \sqrt{\frac{1}{LC}}. \tag{10}$$

and the general solution can be expressed in the form

$$i(t) = Ae^{-\gamma t} \cos \omega_0 t + Be^{-\gamma t} \sin \omega_0 t \tag{11}$$

The constants of integration can be determined by substituting this form of the solution into the original equation and solving for A and B so as to satisfy the initial conditions. In the case where the current is initially zero and $Q_0 = 0$, we have

$$LB\omega_0 = V \tag{12}$$

from which we obtain $B = V/L\omega_0 = V\sqrt{C/L}$. Thus, the complete solution is

$$i(t) = \begin{cases} 0 & \text{for } t < 0 \\ V\sqrt{\frac{C}{L}}e^{-\gamma t} \sin \omega_0 t & \text{for } t > 0. \end{cases} \tag{13}$$