PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 53600

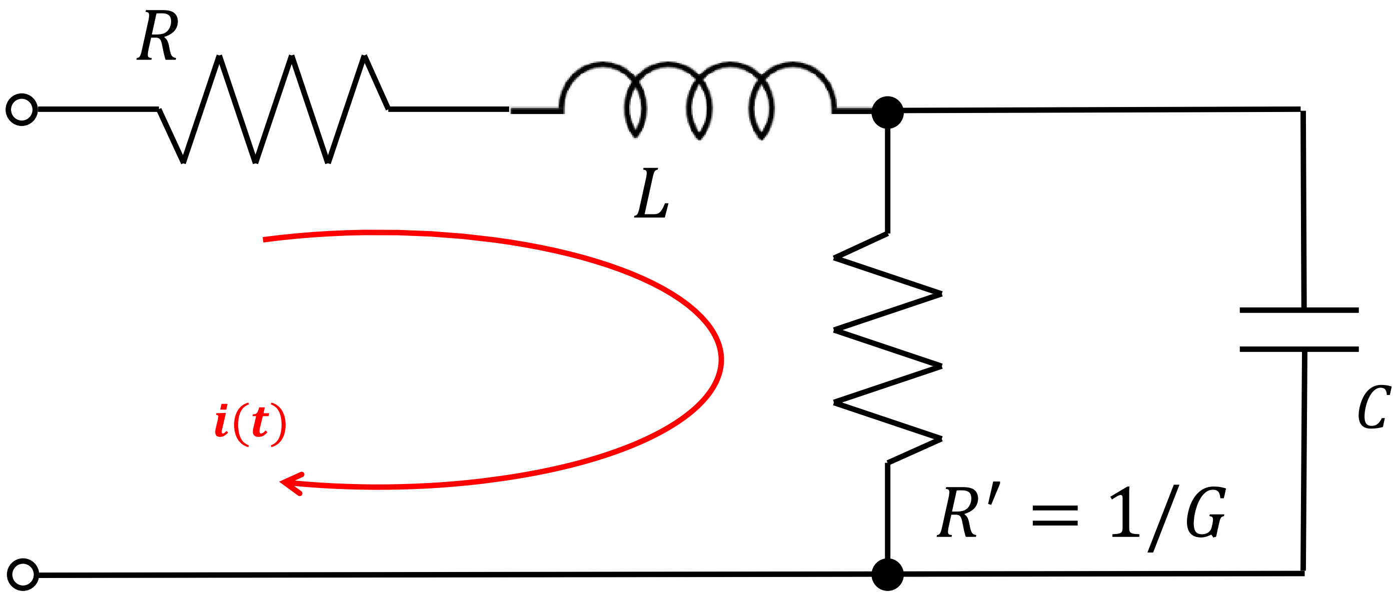
Electronics Techniques for Research

Now in PowerPoint!

Spring 2020 Semester

Prof. Matthew Jones

Transmission Lines



- Total equivalent impedance:

$$Z = R + j\omega L + (G + j\omega C)^{-1}$$
$$= x + \frac{1}{y}$$
$$x = R + j\omega L$$
$$y = G + j\omega C$$

Transmission Lines

- Suppose we had an infinite chain of these lumped circuit elements.
- If we add one more lump, the impedance should not change:

$$Z = R + j\omega L + \left(G + j\omega C + \frac{1}{Z}\right)^{-1}$$

$$= x + \left(y + \frac{1}{Z}\right)^{-1}$$

$$Z = x + \sqrt{\frac{x^2}{4} + \frac{x}{y}}$$

Transmission Lines

- Now suppose that each lump was split into n smaller lumps

$$\left.\begin{array}{l} R \to R/n \\ L \to L/n \end{array}\right\} \quad x \to x/n$$

$$\left.\begin{array}{l} G \to G/n \\ C \to C/n \end{array}\right\} \quad y \to y/n$$

$$Z = \frac{x}{n} + \sqrt{\frac{x^2}{4n^2} + \frac{x/n}{y/n}}$$

$$\lim_{n\to\infty} Z = \sqrt{\frac{x}{y}} = \sqrt{\frac{R + j\omega L}{G + j\omega C}}$$

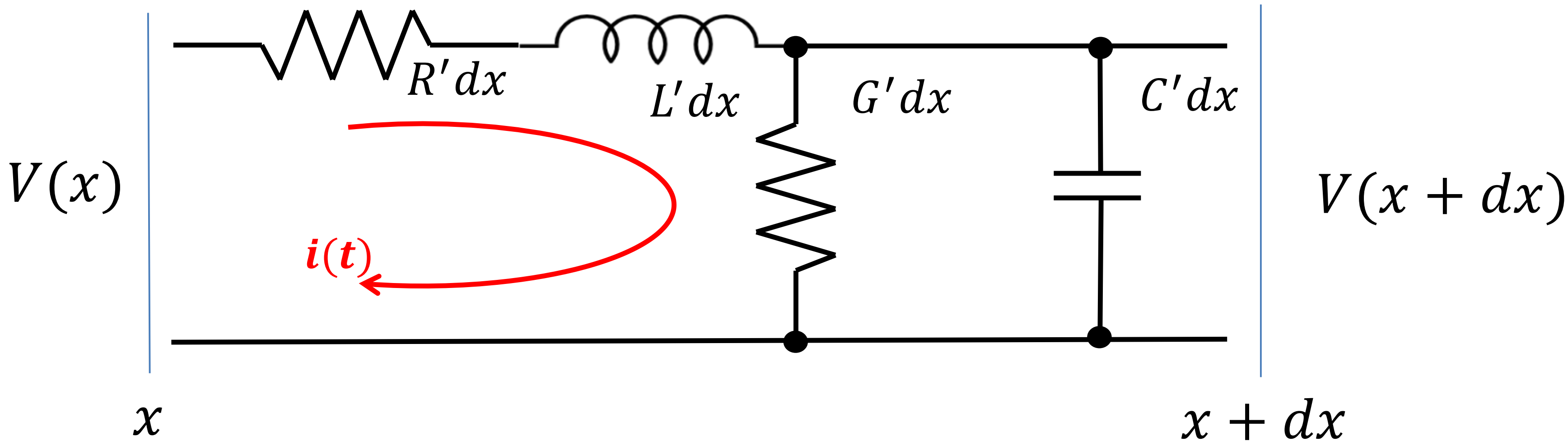
Transmission Lines

- At high frequencies, $\omega L \gg R$ and $\omega C \gg G$

$$Z \to \sqrt{\frac{L'}{C'}}$$

  - Inductance per unit length: L'
  - Capacitance per unit length: C'
  - Resistance per unit length: R'
  - Conductance per unit length: $G' \approx 0$

Transmission Lines



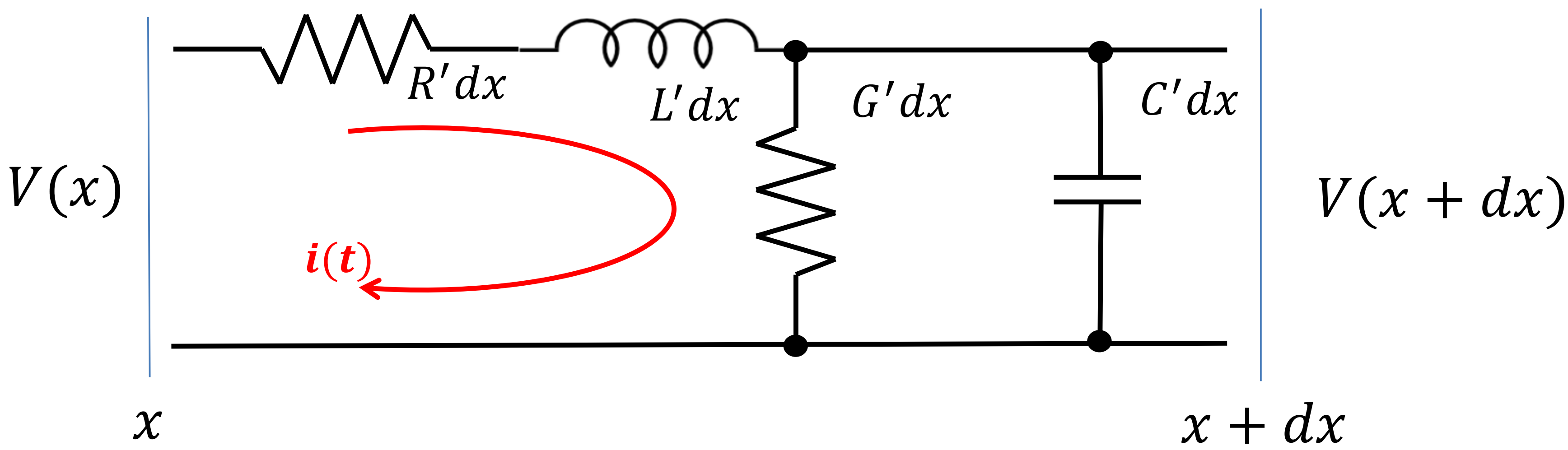
$$I(x+dx) = I(x) - V(x)Y(x)$$

where

$$Y(x) = (G' + j\omega C')dx$$

$$\frac{\partial I}{\partial x} = \frac{I(x+dx) - I(x)}{dx} = -(G' + j\omega C')V(x)$$

Transmission Lines



$$V(x+dx) = V(x) - I(x)X(x)$$

where

$$X(x) = (R' + j\omega L')dx$$

$$\frac{\partial V}{\partial x} = \frac{V(x+dx) - V(x)}{dx} = -(R' + j\omega L')I(x)$$

Transmission Lines

$$\frac{\partial^2 V}{\partial x^2} - \gamma^2 V(x) = 0$$

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$$
$$= \alpha + j\beta$$
$$\beta \approx \omega\sqrt{L'C'}$$

$$V(x) + V_1 e^{\gamma x} + V_2 e^{-\gamma x}$$

Transmission Lines

- Time dependence:

$$V(x,t) + V_1 e^{\alpha x} e^{j(\omega t + \beta x)} + V_2 e^{-\alpha x} e^{j(\omega t - \beta x)}$$

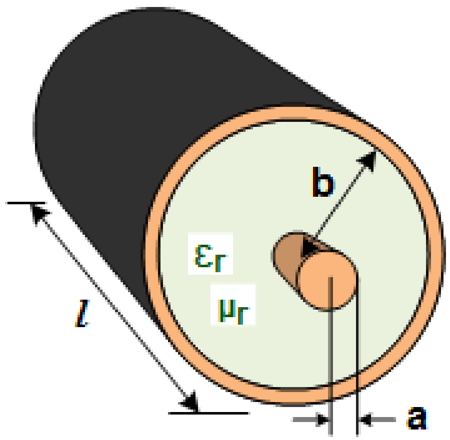
- Speed of wave propagation:

$$\omega t - \beta x = const.$$

$$\omega - \beta \frac{dx}{dt} = 0$$

$$\frac{dx}{dt} = v = \frac{\omega}{\beta} \approx \frac{1}{\sqrt{L'C'}}$$

What are L’ and C’ in practice?



Maxwell’s equation:

$$\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$$

$$\oint_S \boldsymbol{E} \cdot d\boldsymbol{a} = \frac{Q}{\epsilon_0}$$

$$E(r) = \frac{Q}{2\pi\epsilon_0 l} \cdot \frac{1}{r}$$

What are L’ and C’ in practice?

$$V(r) = \int_a^b \boldsymbol{E} \cdot d\boldsymbol{r} = \frac{Q}{2\pi\epsilon_0 l} \log\frac{b}{a}$$

The charge is determined by the capacitance:

$$Q = CV$$

$$C = \frac{2\pi\epsilon_0 l}{\log b/a}$$

If the space between the conductors was filled with a dielectric material then $\epsilon_0 \rightarrow \epsilon = \epsilon_0\epsilon_r$

Capacitance per unit length:

$$C' = \frac{2\pi\epsilon_0\epsilon_r}{\log b/a}$$

What are L’ and C’ in practice?

- Similarly, the inductance per unit length is

$$L' = \frac{\mu_0}{2\pi} \log \frac{b}{a}$$

- Characteristic impedance of a coaxial cable:

$$Z_0 = \sqrt{\frac{L'}{C'}} = \frac{1}{2\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{\log b/a}{\sqrt{\epsilon_r}}$$

$$= (59.97\,\Omega) \frac{\log b/a}{\sqrt{\epsilon_r}}$$

What are L’ and C’ in practice?

- Speed of signal propagation:

$$v = \frac{1}{\sqrt{L'C'}} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c$$

- If a dielectric is used, then

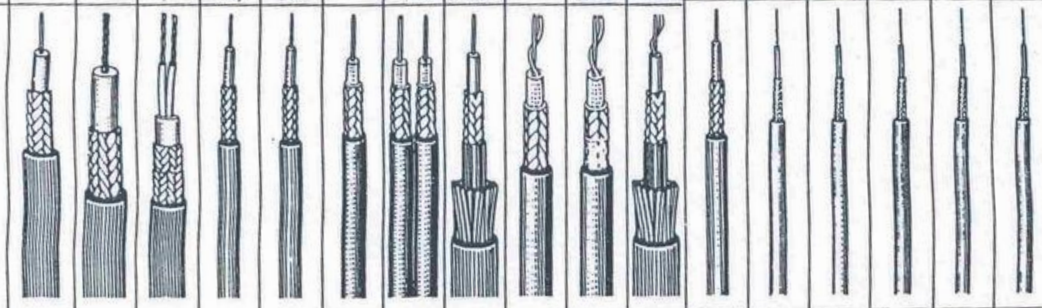
$$v = \frac{c}{\sqrt{\epsilon_r}} < c$$

- A common dielectric material is polyethylene (P.E.) which has $\epsilon_r = 2.25$ ➔ $v \approx \frac{2}{3}c = 20\ \mathrm{cm/ns}$
- Printed circuit boards use FR4/epoxy which has $\epsilon_r \approx 4.2 - 4.5$ ➔ $v \approx \frac{1}{2}c = 15\ \mathrm{cm/ns}$

Coaxial Cables

25 ¢/foot

20 ¢/foot

TRANSRADIO PART NO.	Q 98100	Q 98101	Q 98102	Q 98103	Q 98104	Q 98105	Q 98137	Q 98139	Q 98106	Q 98107	Q 98141	Q 98111	Q 98112	Q 98113	Q 98114	Q 98115	Q 98116
RG TYPE	6A/U	11A/U	22B/U	58 C/U Grey	58 C/U Black	59B/U	59 B/U Twin	59 B/U Armoured	62 A/U	62 A/U Outdoor	62A/U Armoured	142B/U	174U	178B/U	179B/U	180B/U	188A/U
																	
NOM. IMPEDANCE OHMS	75	75	93	50	50	75	75	75	93	93	93	50	50	50	75	95	50
NOM. CAPACITANCE pF/m	67 5	67 5	52	101	101	67 6	67.6	67.6	44.3	44.3	44.3	96.4	101 0	96.4	50.5	50.5	96.4
ATTENUATION db/100M 10MHZ	3 0	1.8	2 8	5.0	5 0	3.5	3.5	3.5	2.9	2.9	2.9	5.0	10	14	8.5	6.0	12
50MHZ	7 0	4.5	6 2	12	12	8.0	8 0	8.0	6.5	6.5	6.5	12 0	24	32	20	14	18
100MHZ	10 0	6.5	9 0	16	16	12	12	12	9.2	9.2	9.2	16	34	46	28	21	37 7
800MHZ	28	22	—	50	50	34	34	34	26	26	26	48	130	150	94	70	90
CONDUCTOR: Material	Cu W SOLID	TiC 7/0 40	2xCu 7/0 40	Cu 19/0 18	Cu 19/0 18	Cu W SOLID	Cu W SOLID	Cu W SOLID	Cu W SOLID	Cu W SOLID	Cu W SOLID	Si.Cu.W. SOLID	Cu.W. 7/0.16	Si.Cu.W 7/0.10	Si.Cu.W 7/0.10	Si.Cu.W 7/0.10	Si.Cu.W 7/0.17
DIA.MM.	0.7	1.2	1.2	0.9	0.9	0.6	0.6	0.6	0.64	0 64	0.64	0.99	0.48	0.305	0.305	0.305	0.50
DIELETRIC: Material	P.E.	P.E.	P.E.	P.E.	P.E	P.E.	P.E.	P.E.	PE+TH	PE+TH	PE+TH	PTFE	PE	PTFE	PTFE	PTFE	PTFE
O/D(NOM.)	4.6	7.2	7.3	3.0	3.0	3.7	3.7	3.7	3.7	3.7	3.7	3.0	1.5	0.86	1.6	2.6	1.5
SCREEN: Material 1st	SiCu	Cu	TiC	TiC	TiC	Cu	Cu	Cu	Cu	Cu	Cu	Si.Cu.	TiC	Si.Cu	Si.Cu	Si.Cu	Si.Cu
2nd	SiCu	—	TiC	—	—	—	—	—	—	—	—	Si.Cu.	—	—	—	—	—
SHEATH: Material	PVC	PVC	PVC	PVC	PVC	PVC	PVC	PVC	PVC	PE	PVC	FEP	PVC	FEP	FEP	FEP	PTFE
O/D(NOM.)	8.4	10.3	10.3	4.9	4 9	6 2	6.2	—	6.2	6.2	—	4.9	2.54	1.9	2.54	3.7	2.8
Weight: Approx KG/KM	119	143	180	43	43	48	96	—	56	57	—	74	11.8	7 4	14.8	28.1	16.2
MIN. BENDING RADIUS	102	114	51	51	51	51	—	—	51	116	—	51	25 4	25.4	25 4	50 8	25 4

Coaxial Cable Properties

- For typical coaxial cable, eg. RG-58

$$C' = 26.4\,\mathrm{pF/foot} = 86.6\,\mathrm{pF/m}$$
$$L' = 0.070\,\mathrm{\mu H/foot} = 0.230\,\mathrm{\mu H/m}$$
$$R' = 17\,\Omega/\mathrm{foot} = 0.056\,\Omega/\mathrm{m}$$

- At what frequency does $|\omega L'| \approx R'$?

$$\omega = \frac{0.056\,\Omega/\mathrm{m}}{0.230\,\mathrm{\mu H/m}} \sim 250\ \mathrm{kHz}$$

- At 1 MHz, $\omega L'$ is four times larger than R'
- At 10 MHz, it is reasonable to ignore R'

Coaxial Cable Properties

- Recall that

$$\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')} = \alpha + j\beta$$

- At low frequencies, say 10 kHz,

$$\gamma \approx \sqrt{j\omega R'C'} = (5.5 \times 10^{-4}\ \mathrm{m}^{-1})\,\frac{(1+\mathrm{j})}{\sqrt{2}}$$

$$\alpha = 3.89 \times 10^{-4}\ \mathrm{m}^{-1}$$

- For example, when $\ell = 100\ \mathrm{m}$, $A' = Ae^{-\alpha\ell}$

$$G = 20\ \log_{10}\frac{A'}{A} = 20\ \log_{10} e^{-\alpha\ell}$$

$$= 20\ (-\alpha\ell)\log_{10}e$$

$$= 0.34\ \mathrm{dB}$$

Coaxial Cable Properties

- At medium frequencies, say 100 MHz,

$$\gamma = \sqrt{(j\omega L')(j\omega C')}$$
$$= j\omega\sqrt{L'C'}$$

- Does this mean that $\alpha \to 0$ as $\omega \to \infty$?
- Skin effect results in resistive losses
- At high frequencies, the electric and magnetic frequencies penetrate conductors with an exponential falloff:

$$E = E_0 e^{-x/\delta}$$
$$\delta = \sqrt{\frac{2}{\omega\mu_0\sigma}}$$

Coaxial Cable Properties

- At high frequencies, the resistance is mostly due to the skin effect:

$$R' \sim \frac{1}{p}\sqrt{\frac{\omega}{\sigma}}$$

where p is the perimeter of the conductor.

- Impedance,

$$Z(\omega) \sim \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

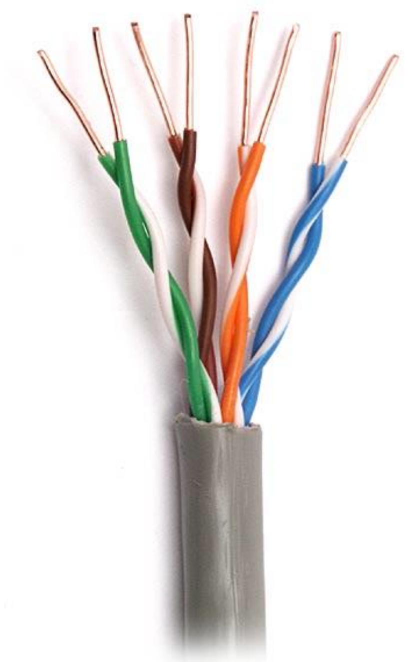
$$\lim_{\omega \to \infty} Z(\omega) = \sqrt{\frac{L'}{C'}}$$

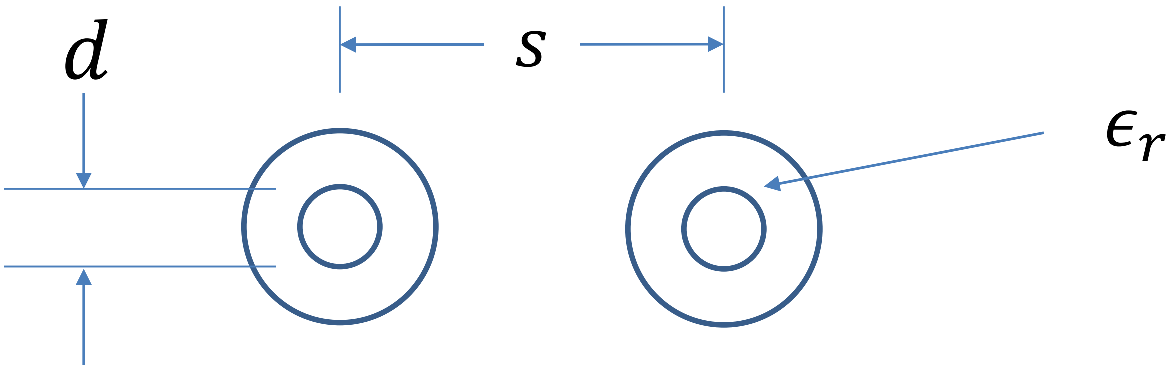
- However, $\gamma = \sqrt{(R' + j\omega L')(G' + j\omega C')}$

$$\lim_{\omega \to \infty} \gamma(\omega) = j\omega\sqrt{L'C'} + \alpha(\omega)$$

$$\alpha(\omega) \approx \sqrt{\omega}$$

Other Transmission Line Configurations

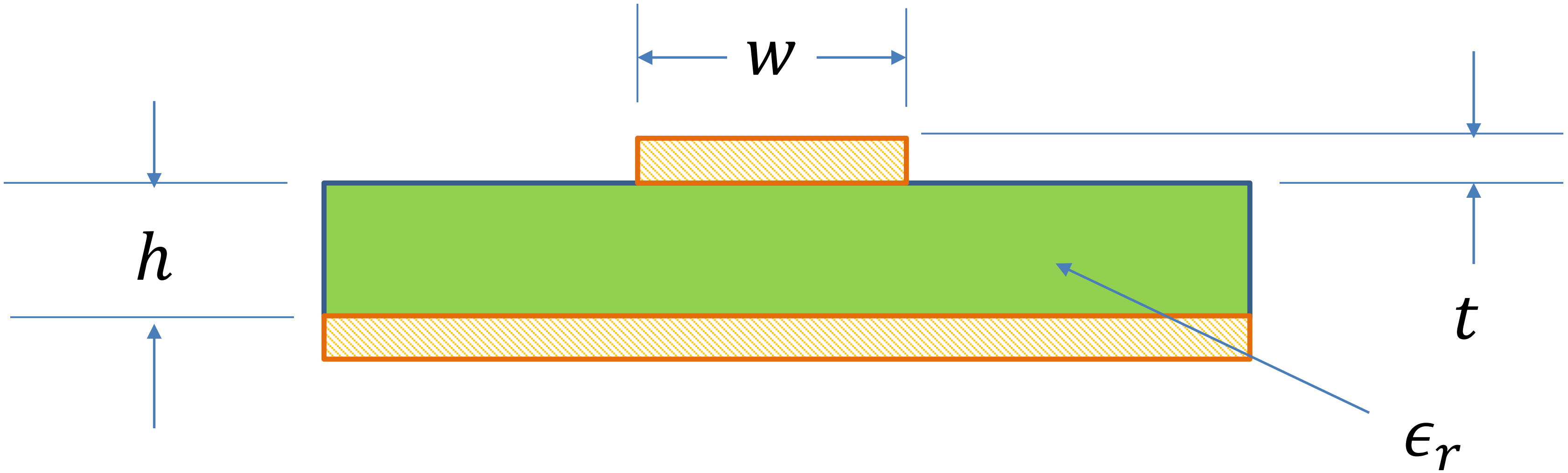




$$Z \approx \frac{120\ \Omega}{\sqrt{\epsilon_r}} \log\left(\frac{2s}{d}\right)$$

- This is called "twisted pair" cable eg. CAT4/5 Ethernet cable

Other Transmission Line Configurations

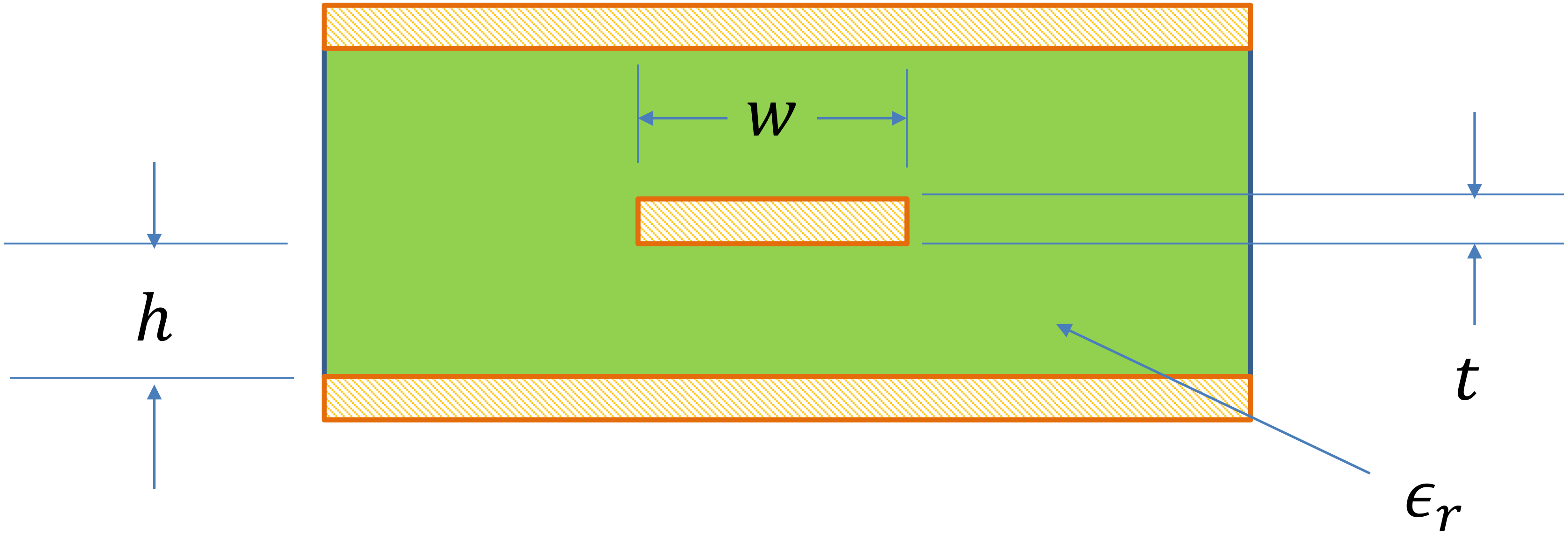


$$Z \approx \frac{87\,\Omega}{\sqrt{1.41+\epsilon_r}} \log\left(\frac{5.98\,h}{t+0.8\,w}\right)$$

ϵ_r is typically about 4.2-4.5

"microstrip" transmission line

Other Transmission Line Configurations



$$Z \approx \frac{60\ \Omega}{\sqrt{\epsilon_r}} \log\left(1.9\ \frac{2h+t}{0.8\,w+t}\right)$$

Embedded microstrip

Transmission Lines

- When do we need to think of a conductor as a transmission line and not just a wire?
- Generally, when the wavelength is shorter than the length of the transmission line
- Example:

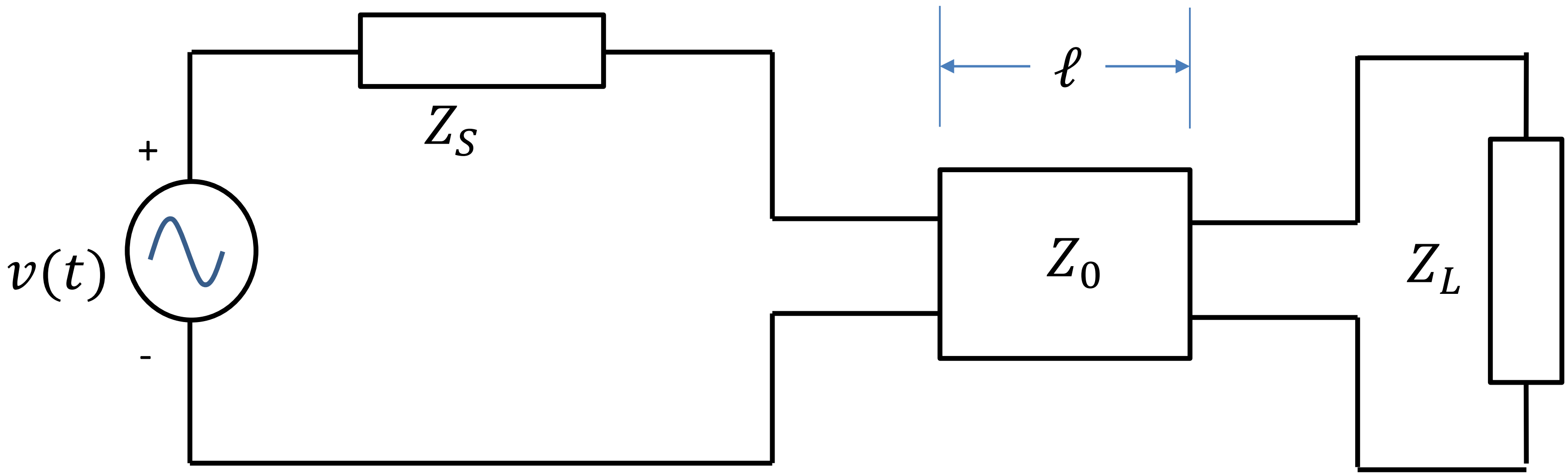
A signal with a fast rise-time of 200 ps:

$$\lambda = v\,\Delta t = \frac{c}{\sqrt{\epsilon_r}}\,\Delta t = 2.8\ \mathrm{cm}$$

A slower signal with $\Delta t \sim 1\ \mathrm{ns}$ over 10 meters of cable:

$$\lambda = v\,\Delta t = 19\ \mathrm{cm} \ll 10\ \mathrm{m}$$

Transmission Lines in Circuits



- Ideal voltage source: $v(t) = V_0 e^{j\omega t}$
- Source impedance: Z_S
- Transmission line impedance: Z_0
- Load impedance: Z_L

Transmission Lines in Circuits

- The general solution to the differential equation in the transmission line:

$$I(x) = Ae^{\gamma x} + Be^{-\gamma x}$$

- Voltage and current in the transmission line are related by

$$V(x) = -\frac{1}{G' + j\omega C'}\frac{\partial I}{\partial x}$$
$$= -\frac{\gamma}{G' + j\omega C'}(Ae^{\gamma x} - Be^{-\gamma x})$$
$$= -Z_0(Ae^{\gamma x} - Be^{-\gamma x})$$

- The constants A and B must make the solution in the transmission line match the boundary conditions at the ends.

Transmission Lines in Circuits

- At the source end, $x = 0$

$$V_0 - I(0)Z_S - V(0) = 0$$
$$I(0) = A + B$$
$$V(0) = -Z_0(A - B)$$
$$V_0 - Z_S(A + B) + Z_0(A - B) = 0$$

- At the load end, $x = \ell$

$$V(\ell) - I(\ell)Z_L = 0$$
$$-Z_0\left(Ae^{\gamma\ell} - Be^{-\gamma\ell}\right) - Z_L\left(Ae^{\gamma\ell} + Be^{-\gamma\ell}\right) = 0$$

- Two equations in two unknowns…

Transmission Lines in Circuits

$$A = \frac{V_0 \Gamma_L e^{-2\gamma\ell}}{(Z_S + Z_0)(1 - \Gamma_S \Gamma_L e^{-2\gamma\ell})}$$

$$B = \frac{V_0}{(Z_S + Z_0)(1 - \Gamma_S \Gamma_L e^{-2\gamma\ell})}$$

$$\left.\begin{aligned} \Gamma_S &= \frac{Z_S - Z_0}{Z_S + Z_0} \\ \Gamma_L &= \frac{Z_L - Z_0}{Z_L + Z_0} \end{aligned}\right\} \text{reflection coefficients}$$

Interpretation

- If V_0 produces a wave travelling to the right, the current induced by this wave in the transmission line is

$$I_0 = \frac{V_0}{Z_0 + Z_S}$$

- This wave propagates down to the end of the transmission line.
- At the far end,

$$I_0' = \frac{V_0}{Z_0 + Z_S} e^{-\gamma \ell}$$

Interpretation

- Some energy is dissipated in the load Z_L but some is reflected back towards the source.
- Reflected component:

$$I_1' = I_0'\Gamma_L = \frac{V_0}{Z_0 + Z_S}\Gamma_L e^{-\gamma\ell}$$

- The reflected wave propagates back towards the source.
- When it arrives,

$$I_1 = I_1' e^{-\gamma\ell} = \frac{V_0}{Z_0 + Z_S}\Gamma_L e^{-2\gamma\ell}$$

Interpretation

- When the reflected wave arrives back at the source, some energy is dissipated in the source impedance, but some is reflected back towards the load.

$$I_2 = I_1 \Gamma_S = \frac{V_0}{Z_0 + Z_S} \Gamma_S \Gamma_L e^{-2\gamma\ell}$$

- This process continues, *ad infinitum*

Interpretation

- Adding up all the components gives:
  - Current flowing in the +x direction:

$$I_+(x) = \frac{V_0}{Z_0 + Z_S} \frac{e^{-\gamma x}}{1 - \Gamma_S \Gamma_L e^{-2\gamma \ell}}$$

  - Current flowing in the –x direction:

$$I_-(x) = \frac{V_0}{Z_0 + Z_S} \frac{\Gamma_L e^{-2\gamma \ell} e^{\gamma x}}{1 - \Gamma_S \Gamma_L e^{-2\gamma \ell}}$$

- If $Z_L = Z_0$ then no energy is reflected from the load.
- If $Z_S = Z_0$ then no energy is reflected from the source.