Purdue University Department of Physics

Physics 53600

Electronics Techniques for Research

Now in PowerPoint!

Spring 2020 Semester

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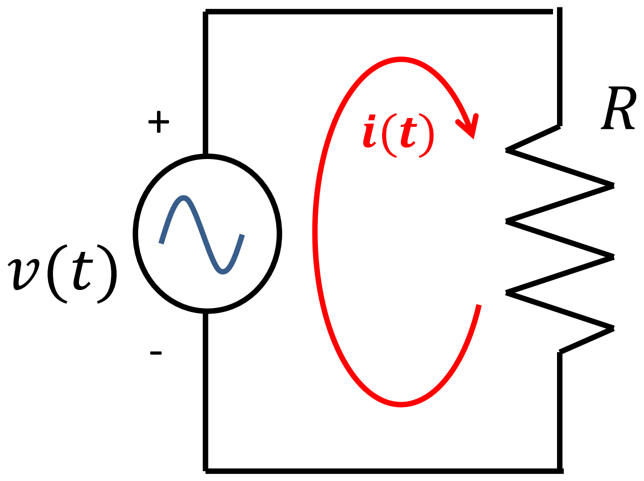
AC Circuits

- So far we have studied circuits with only DC voltage sources
- AC voltage sources are also very common
- There are two types of solutions:
  - Transient response (immediately after turned on)
  - Steady state response (after damped transients)
- Both can be analyzed using the same techniques we have previously used
  - The steady state response is a bit easier to analyze

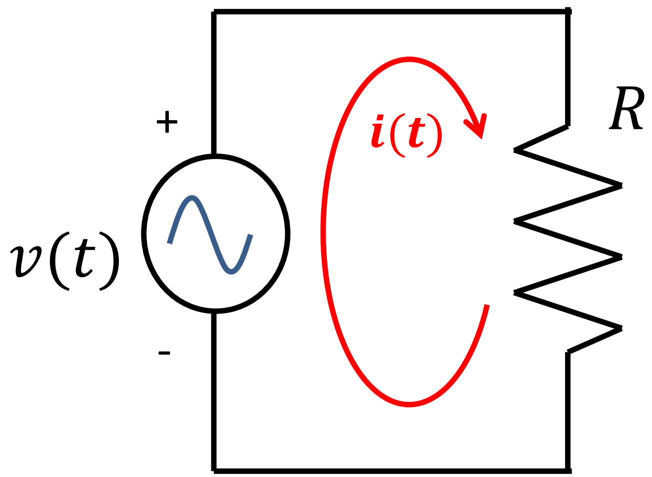
A Very Simple Circuit



- The phase of the voltage source at $t = 0$ needs to be specified but usually we are free to pick it for convenience.

$$v(t) = V_0 \cos \omega t$$

A Very Simple Circuit

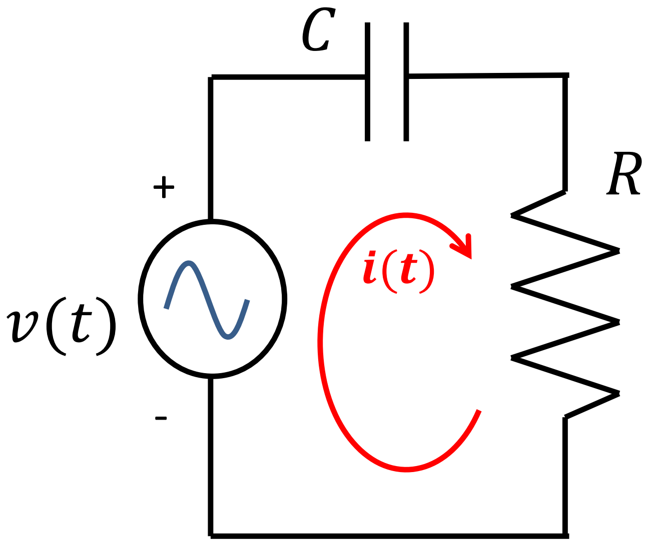


- The current is simply given by Ohm's law

$$i(t) = \frac{v(t)}{R} = \frac{V_0}{R}\cos\omega t$$

- The current has the same phase as the voltage

A Slightly Less Simple Circuit



$$v(t) - \frac{1}{C}\int_0^t i(\tau)d\tau - i(t)R = 0$$

$$\frac{di}{dt} + \frac{i(t)}{RC} = \frac{1}{R}\frac{dv}{dt} = -\frac{V_0\omega}{R}\sin\omega t$$

A Slightly Less Simple Circuit

- We already solved the homogeneous equation

$$\frac{di}{dt} + \frac{i(t)}{RC} = 0$$

$$i_0(t) = A\, e^{-t/RC}$$

- Try to find a solution to the non-homogeneous equation. Maybe it is of the form

$$i(t) = B\cos\omega t + D\sin\omega t$$

$$-\omega B\sin\omega t + \omega D\cos\omega t + \frac{B}{RC}\cos\omega t + \frac{D}{RC}\sin\omega t$$

$$= -\frac{V_0\omega}{R}\sin\omega t$$

A Slightly Less Simple Circuit

$$-\omega B \sin \omega t + \omega D \cos \omega t + \frac{B}{RC} \cos \omega t + \frac{D}{RC} \sin \omega t$$
$$= -\frac{V_0 \omega}{R} \sin \omega t$$

- Equate coefficients in front of $\sin \omega t$ and $\cos \omega t$

$$D = -\frac{B}{\omega RC}$$
$$-B - \frac{B}{\omega^2 R^2 C^2} = -\frac{V_0}{R}$$
$$B = \frac{V_0}{R} \frac{1}{1 + 1/\omega^2 R^2 C^2}$$
$$D = -\frac{V_0}{R} \frac{1/\omega RC}{1 + 1/\omega^2 R^2 C^2}$$

A Slightly Less Simple Circuit

$$i(t) = B\cos\omega t + D\sin\omega t$$

$$= \frac{V_0}{R}\frac{1}{1+\frac{1}{\omega^2R^2C^2}}\left(\cos\omega t - \frac{1}{\omega RC}\sin\omega t\right)$$

$$= \frac{V_0}{R}\frac{1}{\sqrt{1+\frac{1}{\omega^2R^2C^2}}}(\cos\varphi\cos\omega t - \sin\varphi\sin\omega t)$$

$$i(t) = \frac{V_0}{R}\frac{1}{\sqrt{1+\frac{1}{\omega^2R^2C^2}}}\cos(\omega t - \varphi)$$

$$\varphi = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

A Slightly Less Simple Circuit

- The complete solution to the inhomogeneous equation is:

$$i(t) = A\,e^{-t/RC} + \frac{V_0}{R}\frac{1}{\sqrt{1+\frac{1}{\omega^2R^2C^2}}}\cos(\omega t-\varphi)$$

- Initial conditions:

$$i(0) = 0$$

$$A = -\frac{V_0}{R}\frac{1}{\sqrt{1+\frac{1}{\omega^2R^2C^2}}}\cos\varphi$$

Transient response

Steady state response

$$i(t) = \frac{V_0}{R}\frac{1}{\sqrt{1+\frac{1}{\omega^2R^2C^2}}}\left(\cos(\omega t-\varphi) - e^{-t/RC}\cos\varphi\right)$$

AC Circuit Analysis

- Frequently, we don't care about the transient part of the solution.
  - For example, $t \gg RC$
- Then, we only need to consider solutions of the form

$$i(t) = A\ \cos(\omega t - \varphi)$$

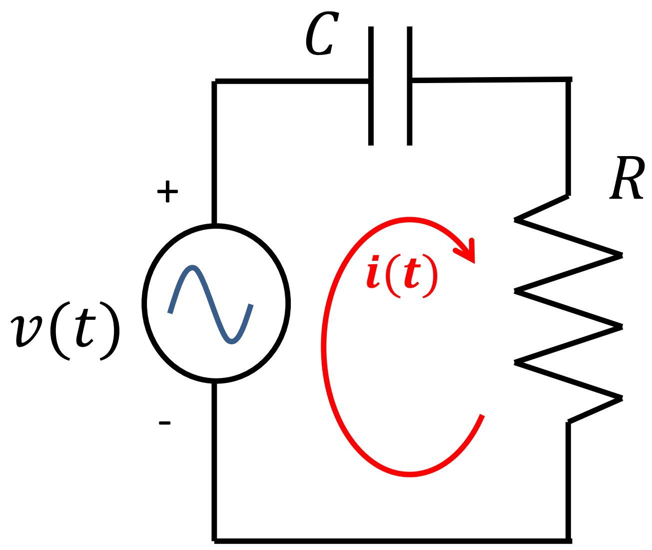
- We can also express this in terms of complex numbers:

$$v(t) = V_0 \mathrm{Re}\left[\, e^{j\omega t}\right]$$

$$i(t) = \mathrm{I}_0 \mathrm{Re}\left[e^{j(\omega t - \varphi)}\right] = \mathrm{Re}[c e^{j\omega t}]$$

$$c = I_0 e^{-j\varphi} \in \ \mathbb{C}$$

AC Circuit Analysis



$$v(t) - \frac{1}{C}\int_0^t i(\tau)d\tau - i(t)R = 0$$

$$V_0 e^{j\omega t} + \left(\frac{j}{\omega C} - R\right) c e^{j\omega t} = 0$$

$$c = \frac{V_0}{R - j/\omega C} = \frac{V_0/R}{1 - j/\omega RC} = \frac{\frac{V_0}{R}\left(1 + \frac{j}{\omega RC}\right)}{1 + \frac{1}{\omega^2 R^2 C^2}}$$

AC Circuit Analysis

$$c = \frac{\frac{V_0}{R}\left(1 + \frac{j}{\omega RC}\right)}{1 + \frac{1}{\omega^2 R^2 C^2}}$$

$$i(t) = \frac{V_0}{R} \frac{1}{\sqrt{1 + \frac{1}{\omega^2 R^2 C^2}}} (\cos\varphi \cos\omega t - \sin\varphi \sin\omega t)$$

$$i(t) = \frac{V_0}{R} \frac{1}{\sqrt{1 + \frac{1}{\omega^2 R^2 C^2}}} \cos(\omega t - \varphi)$$

$$\varphi = \tan^{-1}\left(\frac{1}{\omega RC}\right)$$

AC Circuit Analysis

- Voltage sources are of the form

$$v(t) = V_0 \cos \omega t$$

- Resulting currents are of the form

$$i(t) = I_0 \cos(\omega t - \varphi)$$

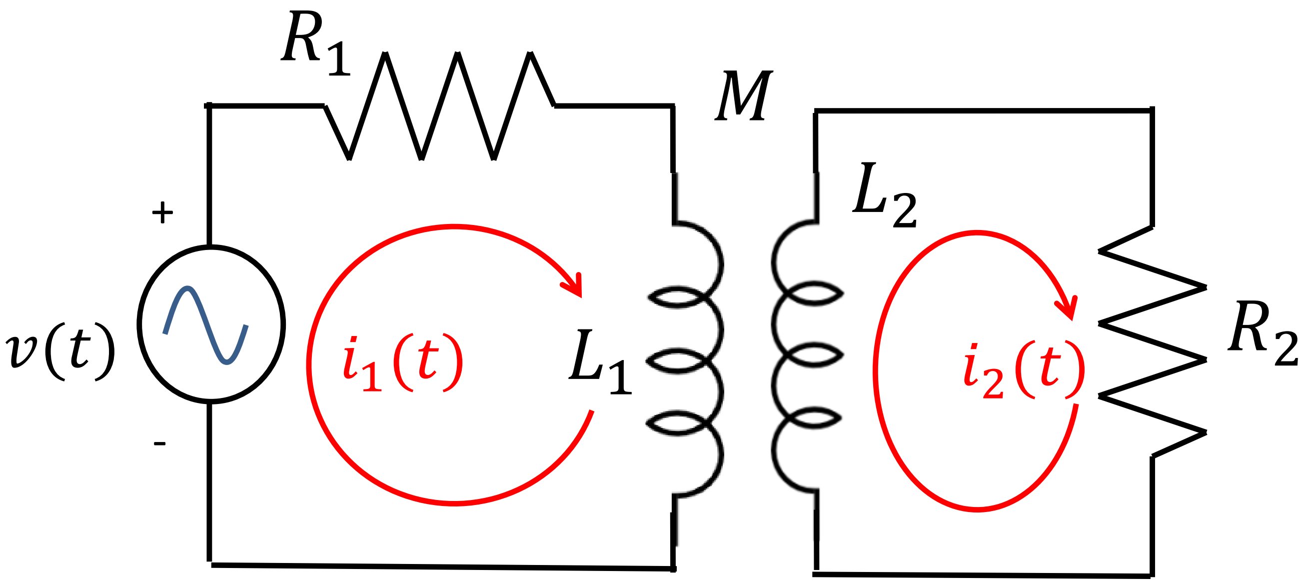
- The differential equation is transformed into an algebraic equation

Impedance

$$Z_R = R$$
$$Z_C = -\frac{j}{\omega C} = j\,X_C$$
$$Z_L = j\,\omega L = j\,X_L$$

- Impedance: Z
- Resistance: R (real part of impedance)
- Reactance: X (imaginary part of impedance)

More Complicated Circuit



$$V_0 - i_1 R_1 - Z_{L1} i_1 + Z_M i_2 = 0$$
$$-i_2 R_2 - Z_{L2} i_2 + Z_M i_1 = 0$$
$$\begin{pmatrix} R_1 + Z_{L1} & -Z_M \\ -Z_M & R_2 + Z_{L2} \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \end{pmatrix} = \begin{pmatrix} V_0 \\ 0 \end{pmatrix}$$

More Complicated Circuit

$$i_1 = \frac{\begin{vmatrix} V_0 & -Z_M \\ 0 & R_2 + Z_{L2} \end{vmatrix}}{\begin{vmatrix} R_1 + Z_{L1} & -Z_M \\ -Z_M & R_2 + Z_{L2} \end{vmatrix}} = \frac{V_0(R_2 + Z_{L2})}{(R_1 + Z_{L1})(R_2 + Z_{L2}) - Z_M^2}$$

$$i_2 = \frac{\begin{vmatrix} R_1 + Z_{L1} & V_0 \\ -Z_M & 0 \end{vmatrix}}{\begin{vmatrix} R_1 + Z_{L1} & -Z_M \\ -Z_M & R_2 + Z_{L2} \end{vmatrix}} = \frac{-V_0 Z_M}{(R_1 + Z_{L1})(R_2 + Z_{L2}) - Z_M^2}$$

- Now we just need to simplify using complex math.
- This is why COMPLEX was a built-in type in programming languages like FORTRAN

More Complicated Circuit

$$i_2 = \frac{-V_0(jX_M)}{R_1R_2 - X_{L1}X_{L2} + X_M^2 + j(R_1X_{L2} + R_2X_{L1})}$$

- Let’s assume perfect coupling:

$$X_{L1}X_{L2} = X_M^2$$

$$i_2 = \frac{-V_0(jX_M)}{R_1R_2 + j(R_1X_{L2} + R_2X_{L1})}$$

$$= \frac{-V_0X_M(R_1X_{L2} + R_2X_{L1}) - jV_0X_MR_1R_2}{R_1^2R_2^2 + (R_1X_{L2} + R_2X_{L1})^2}$$

More Complicated Circuit

$$i_2 = \frac{-V_0 X_M (R_1 X_{L2} + R_2 X_{L1}) - jV_0 X_M R_1 R_2}{R_1^2 R_2^2 + (R_1 X_{L2} + R_2 X_{L1})^2}$$

- Magnitude of current:

$$|i_2| = \frac{V_0 X_M}{\sqrt{R_1^2 R_2^2 + (R_1 X_{L2} + R_2 X_{L1})^2}}$$

- Phase:

$$\varphi = \tan^{-1}\left(\frac{R_1 R_2}{R_1 X_{L2} + R_2 X_{L1}}\right)$$

More Complicated Circuit

- Common assumptions:

$$X_{L1} \gg R_1$$
$$X_{L2} \gg R_2$$

  – Easy to achieve at high frequencies since $X_L = \omega L$

- Magnitude of current:

$$|i_2| = \frac{V_0 X_M}{R_1 X_{L2} + R_2 X_{L1}} = \frac{V_0}{R_1\sqrt{\frac{X_{L2}}{X_{L1}}} + R_2\sqrt{\frac{X_{L1}}{X_{L2}}}}$$

- Phase:

$$\varphi = \tan^{-1}(\text{small}) \approx 0$$

Impedance Matching

- The power delivered to the load is

$$P = \frac{|i_2|^2}{R_2}$$

- This is maximized when

$$\frac{X_{L1}}{X_{L2}} = \frac{N_1^2}{N_2^2} = \frac{R_1}{R_2}$$

- This will probably be an exercise to be assigned on Thursday.