Assignment # 1.

1. (a) We know that there is no net free charge inside the resistor even when current is flowing. From the axial symmetry we know that the electric field will be directed along the axis of the resistor.

Poisson's equation is then written

$$\frac{\partial E}{\partial x} = 0$$

and therefore E must be constant.

(b) The current density is related to the conductivity:

$$J = \sigma E$$

where by symmetry and homogeneity of the material, J will be in the x-direction.

(c) I really should have asked this question first, because we use the fact that the charge density is zero to solve for E.

2. The potential difference across the resistor is

(a)
$$\Delta V = \int_0^L E(x)\,dx$$
$$= \int_0^{L/2} E_1\,dx + \int_{L/2}^{L} E_2\,dx$$
$$= \frac{L}{2}\left(E_1 + E_2\right)$$

But, jumping ahead to part (b), the current density will be constant everywhere in the resistor.

Therefore, $J = \sigma_1 E_1 = \sigma_2 E_2$.

Hence, $E_2 = \frac{\sigma_1}{\sigma_2} E_1$

So $\Delta V = \frac{L}{2}\left(1 + \frac{\sigma_1}{\sigma_2}\right) E_1$
$$= \frac{L}{2}(\sigma_1 + \sigma_2)\frac{E_1}{\sigma_2}$$

So $E_1 = \frac{2\Delta V}{L} \frac{\sigma_2}{\sigma_1 + \sigma_2}$

Likewise, $E_2 = \frac{2\Delta V}{L} \frac{\sigma_1}{\sigma_1 + \sigma_2}$.

(b) As discussed in part (a) the current density is constant,

$$J = \sigma_1 E_1 = \frac{2\Delta V}{L} \frac{\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} = \sigma_2 E_2 .$$

(c) At the interface between the materials we can use Gauss' law to determine the charge density:

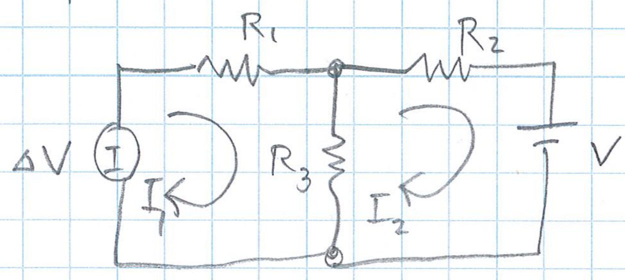
$$\int_V \nabla \cdot \vec{E}\, dv = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow (-E_1 + E_2) A = \frac{Q}{\varepsilon_0}$$

Hence, the charge density at the interface is

$$\frac{Q}{A} = \varepsilon_0 (E_2 - E_1)$$

3. This is still a linear system but now we need to solve for one voltage source (potential difference) and one current.



Loop 1
$$\begin{cases} -I_1R_1 - (I_1 - I_2)R_3 + \Delta V = 0 \\ I_1(R_1 + R_3) - I_2R_3 - \Delta V = 0 \\ \dfrac{I_2R_3}{R_1+R_3} + \dfrac{\Delta V}{R_1+R_3} = I \end{cases}$$

Loop 2
$$\begin{cases} I_2R_2 - (I_2 - I_1)R_3 - V = 0 \\ I_2(R_2 + R_3) = V - IR_3 \end{cases}$$

(a)
$$\begin{pmatrix} 1/R_1+R_3 & R_3/R_1+R_3 \\ 0 & R_2+R_3 \end{pmatrix}\begin{pmatrix} \Delta V \\ I_2 \end{pmatrix} = \begin{pmatrix} I \\ V - IR_3 \end{pmatrix}$$

(b)
$$\Delta V = \frac{\begin{vmatrix} I & R_3/R_1+R_3 \\ V-IR_3 & R_2+R_3 \end{vmatrix}}{\begin{vmatrix} 1/R_1+R_3 & R_3/R_1+R_3 \\ 0 & R_2+R_3 \end{vmatrix}}$$

$$= \frac{I(R_2+R_3) - (R_3V - IR_3^2)/(R_1+R_3)}{\left(\dfrac{R_2+R_3}{R_1+R_3}\right)}$$

$$= (V + IR_2)\frac{R_3}{R_2+R_3}$$

(c)

$$I_2 = \frac{\begin{vmatrix} 1/R_1 + R_3 & I \\ 0 & V - IR_3 \end{vmatrix}}{\begin{vmatrix} 1/R_1 + R_3 & R_3/R_1 + R_3 \\ 0 & R_2 + R_3 \end{vmatrix}}$$

$$= \left(\frac{V - IR_3}{R_1 + R_2} \right) \Big/ \left(\frac{R_2 + R_3}{R_1 + R_2} \right)$$

$$= \frac{V - IR_3}{R_2 + R_3}$$