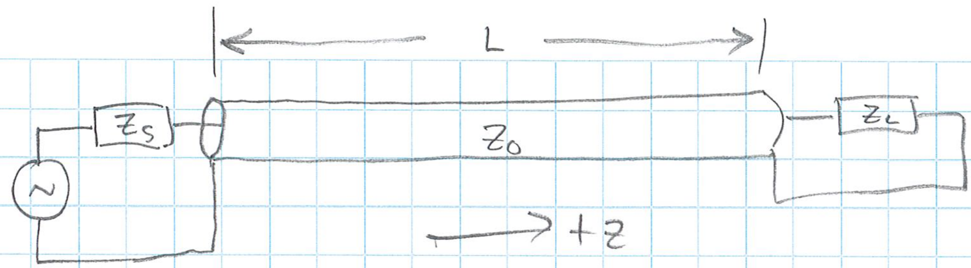
Assignment #6

1.



(a) The reflection coefficient at the load is

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0}$$

where Z_0 is the characteristic impedance of the transmission line.

(b) Likewise, the reflection coefficient at the source is

$$\Gamma_S = \frac{Z_S - Z_0}{Z_S + Z_0}$$

but because $Z_S = Z_0$, $\Gamma_S = 0$.

(c) The wave propagating in the $+z$ direction can be written

$$V_i(z,t) = V_0 \cos(\omega t - kz)$$

where $k = \omega / v$.

(d) The voltage of the reflected wave at the load is

$$V_r(L,t) = \Gamma_L V_0 \cos(\omega t - kL).$$

This wave propagates in the $-z$ direction and is described thus:

$$\begin{aligned} V_r(z,t) &= \Gamma_L V_0 \cos(\omega t - kL - k(L - z)) \\ &= \Gamma_L V_0 \cos(\omega t - 2kL + kz) \end{aligned}$$

(e) At the source, the measured voltage would be

$$V(t) = V_i(0,t) + V_r(0,t)$$
$$= V_0 \cos(\omega t) + r_L V_0 \cos(\omega t - 2kL)$$

There are other ways to express this, but an understanding of the VSWR can be obtained with this form.

(f) The maximum amplitude occurs at those frequencies where there is constructive interference:

$$V_{max} = V_0 + |r_L| V_0$$

The absolute value is necessary because r_L could be positive or negative.

The minimum amplitude is

$$V_{min} = V_0 - |r_L| V_0$$

so the VSWR is

$$VSWR = \frac{1 + |r_L|}{1 - |r_L|}$$

If r_L is positive, then

$$VSWR = \frac{1 + r_L}{1 - r_L} = \frac{Z_L + Z_0 + Z_L - Z_0}{Z_L + Z_0 - Z_L + Z_0} = \frac{Z_L}{Z_0}$$

So $Z_L = VSWR \cdot Z_0$.

If r_L is negative, as in $Z_L < Z_0$,

$$VSWR = \frac{1 - r_L}{1 + r_L} = \frac{Z_L + Z_0 - (Z_L - Z_0)}{Z_L + Z_0 + (Z_L - Z_0)}$$

$$= \frac{Z_0}{Z_L}$$

So $Z_L = \dfrac{Z_0}{VSWR}$.

Thus, there is an ambiguity in the determination of Z_L.

2. The wave equation is

$$\nabla^2 \psi - \frac{1}{v^2}\frac{\partial^2 \psi}{\partial t^2} = 0$$

and we can construct solutions of the form

$$\psi(\vec{x}, t) = A\, f_x(x)\, f_y(y)\, f_z(z) \cos \omega t$$

where $f_x(x)$, $f_y(y)$ and $f_z(z)$ satisfy the boundary conditions

$$f_x(0) = f_x(L_x) = 0$$

$$f_y(0) = f_y(L_y) = 0$$

$$f_z(0) = f_z(L_z) = 0$$

One set of functions that satisfies these boundary conditions is

$$f_x(x) = \sin\left(\frac{\pi n_1 x}{L_x}\right)$$

$$f_y(y) = \sin\left(\frac{\pi n_2 y}{L_y}\right)$$

$$f_z(z) = \sin\left(\frac{\pi n_3 z}{L_z}\right)$$

where n_1, n_2 and n_3 are positive integers.

Thus, $\nabla^2 \psi = \left[-\left(\frac{\pi n_1}{L_x}\right)^2 - \left(\frac{\pi n_2}{L_y}\right)^2 - \left(\frac{\pi n_3}{L_z}\right)^2\right] \psi$

$$\frac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$$

So $\nabla^2\psi - \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$

$$= -\left(\left(\frac{\pi n_1}{L_x}\right)^2 + \left(\frac{\pi n_2}{L_y}\right)^2 + \left(\frac{\pi n_3}{L_z}\right)^2 - \frac{\omega^2}{v^2} \right)\psi$$

$$= 0$$

Thus, $\omega = \pm v \sqrt{\left(\frac{\pi n_1}{L_x}\right)^2 + \left(\frac{\pi n_2}{L_y}\right)^2 + \left(\frac{\pi n_3}{L_z}\right)^2}$

for any choice of the positive integers n_1, n_2, n_3.

3. One way to analyze this problem is to consider the sheet as being composed of many thinner sheets, each with thickness Δx in which the index of refraction is approximately constant.

If the sheets are numbered $1, \ldots, N$ then

$$n_j = n(j\Delta x)$$

From Snell's law,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2$$

where θ_2 in this case is the angle inside the first slice. Applying the same argument to the interface between the first and second slices implies that

$$n_2 \sin\theta_2 = n_3 \sin\theta_3$$

But from the previous equation,

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 = n_3 \sin\theta_3$$

This argument can be repeated for all the slices to conclude that

$$n_1 \sin\theta_1 = n_{N+1} \sin\theta_{N+1}$$

But if the medium is the same on both sides, $n_1 = n_{N+1}$ and so

$$\sin\theta_1 = \sin\theta_{N+1} \Rightarrow \theta_1 = \theta_{N+1}$$

N was arbitrary so we can let $N \to \infty$ without loss of generality.