PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

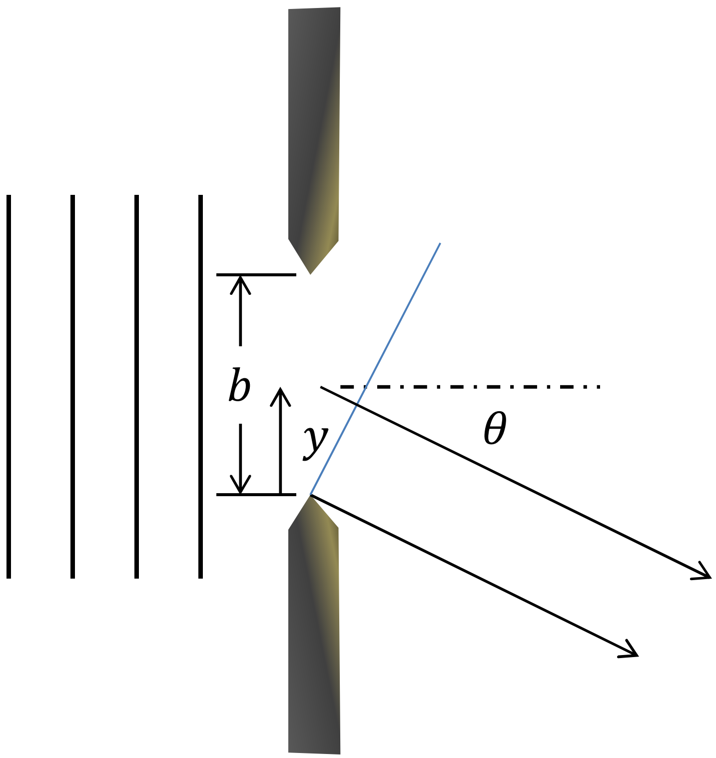
Waves & Oscillations

Lecture 38 – Fresnel Diffraction

Spring 2014 Semester

Matthew Jones

Fraunhofer Diffraction

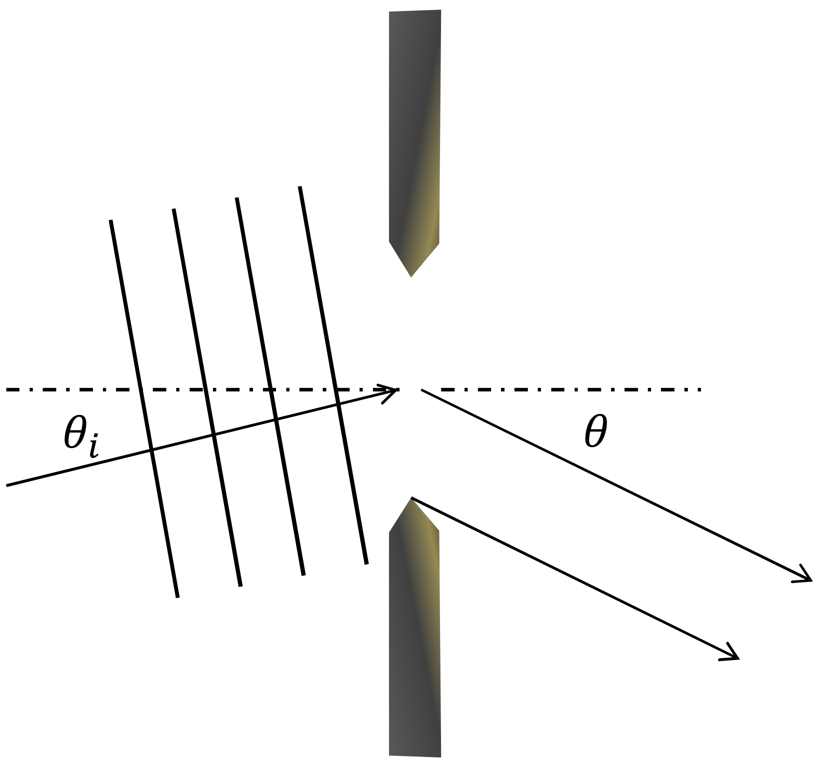


Fraunhofer diffraction:

- The phase varies linearly across the aperture
- The intensity of light arriving from each part of the aperture is equal

$$dE = \frac{\mathcal{E}_L e^{ik\boldsymbol{y}\sin\theta}dy}{\boldsymbol{R}}$$

Fraunhofer Diffraction



Fraunhofer diffraction:

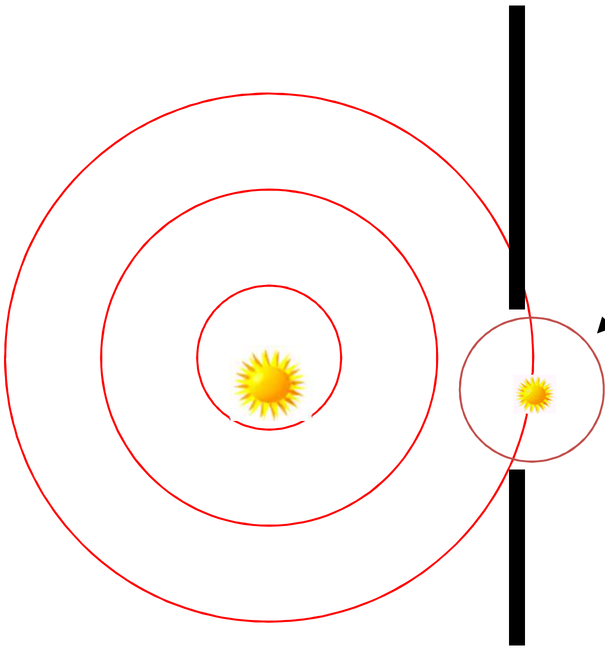
- The phase varies linearly across the aperture
- The intensity of light arriving from each part of the aperture is equal

$$dE = \frac{\mathcal{E}_L e^{iky(\sin\theta - \sin\theta_i)} dy}{R}$$

Huygens-Fresnel Principle

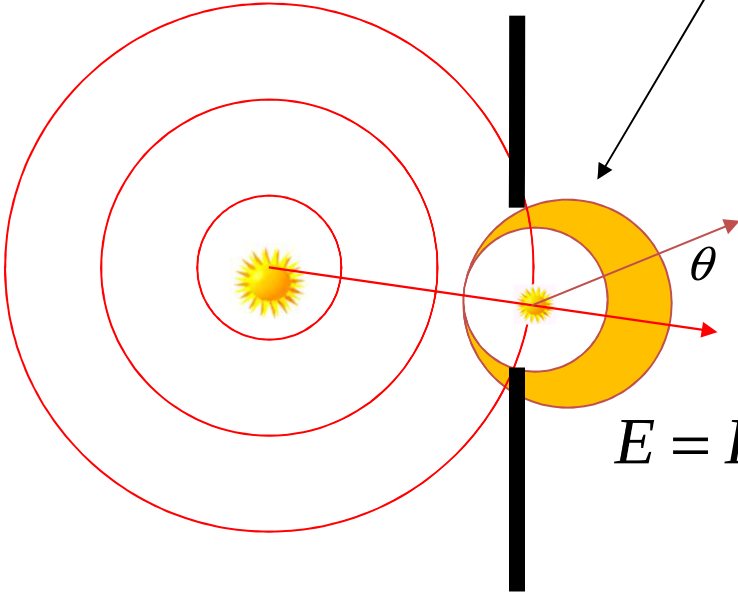
- Each point on a wave front is a source of spherical waves that are in phase with the incident wave.
- The light at any point in the direction of propagation is the sum of all such spherical waves, taking into account their relative phases and path lengths.
- *The secondary spherical waves are preferentially emitted in the forward direction.*
- Fresnel presented a very different way of thinking about the propagation and diffraction of light.
  - The details might be the subject of extensive debate
  - It relies completely on the wave nature of light
  - The predictions were confirmed by experiment

Huygens-Fresnel Principle



Huygens: source of spherical wave

Problem: it must also go backwards, but that is not observed in experiment



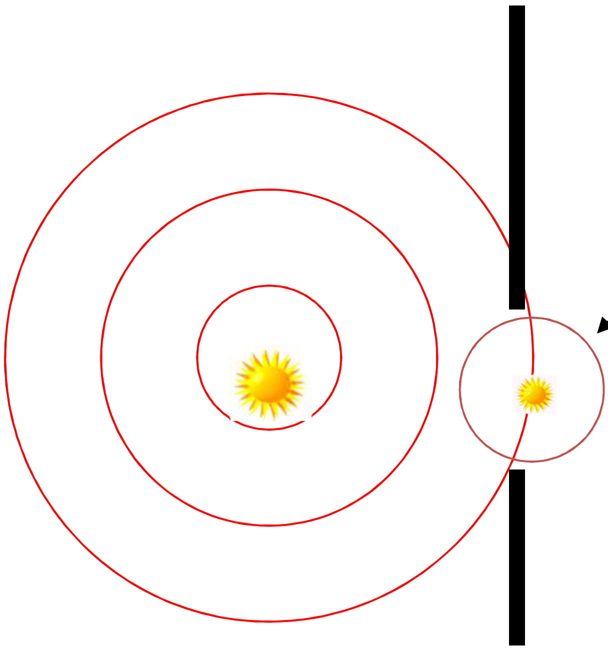
Intensity of the spherical wavelet depends on direction. Fresnel supposed that…

$$K(\theta) \to 0 \text{ as } \theta \to \frac{\pi}{2}$$

inclination factor

$$E = K(\theta)\frac{E}{r}\cos(\omega t - kr + \xi)$$

Huygens-Fresnel Principle



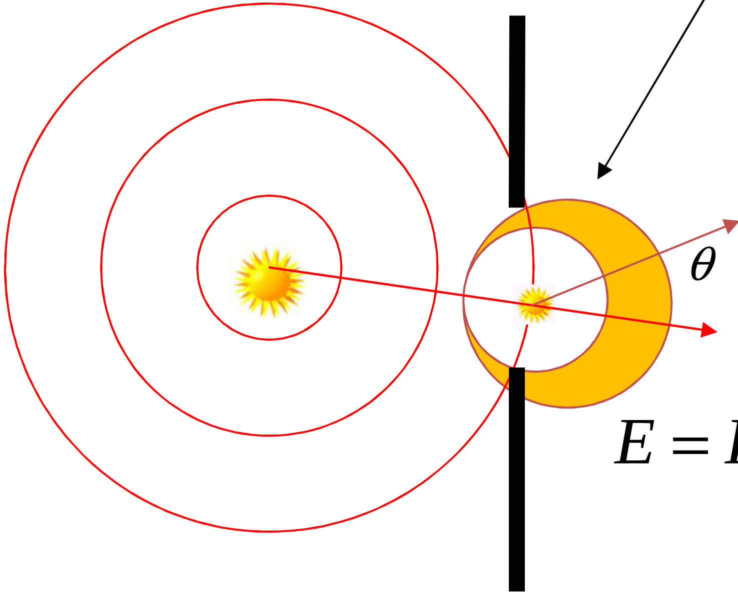
Huygens: source of spherical wave

Problem: it must also go backwards, but that is not observed in experiment

Intensity of the spherical wavelet depends on direction…

$$K(\theta) = \frac{1}{2}(1 + \cos\theta)$$

inclination factor

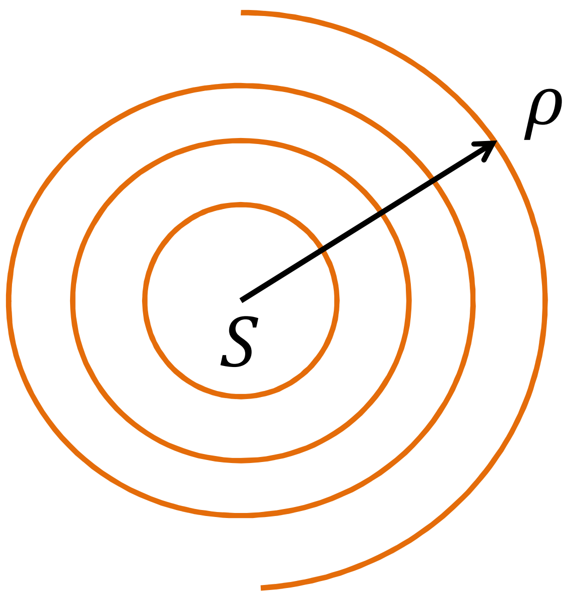


$$E = K(\theta)\frac{E}{r}\cos(\omega t - kr + \xi)$$

Propagation of Spherical Waves

- Consider a spherical wave emitted from a source S at time $t = 0$.

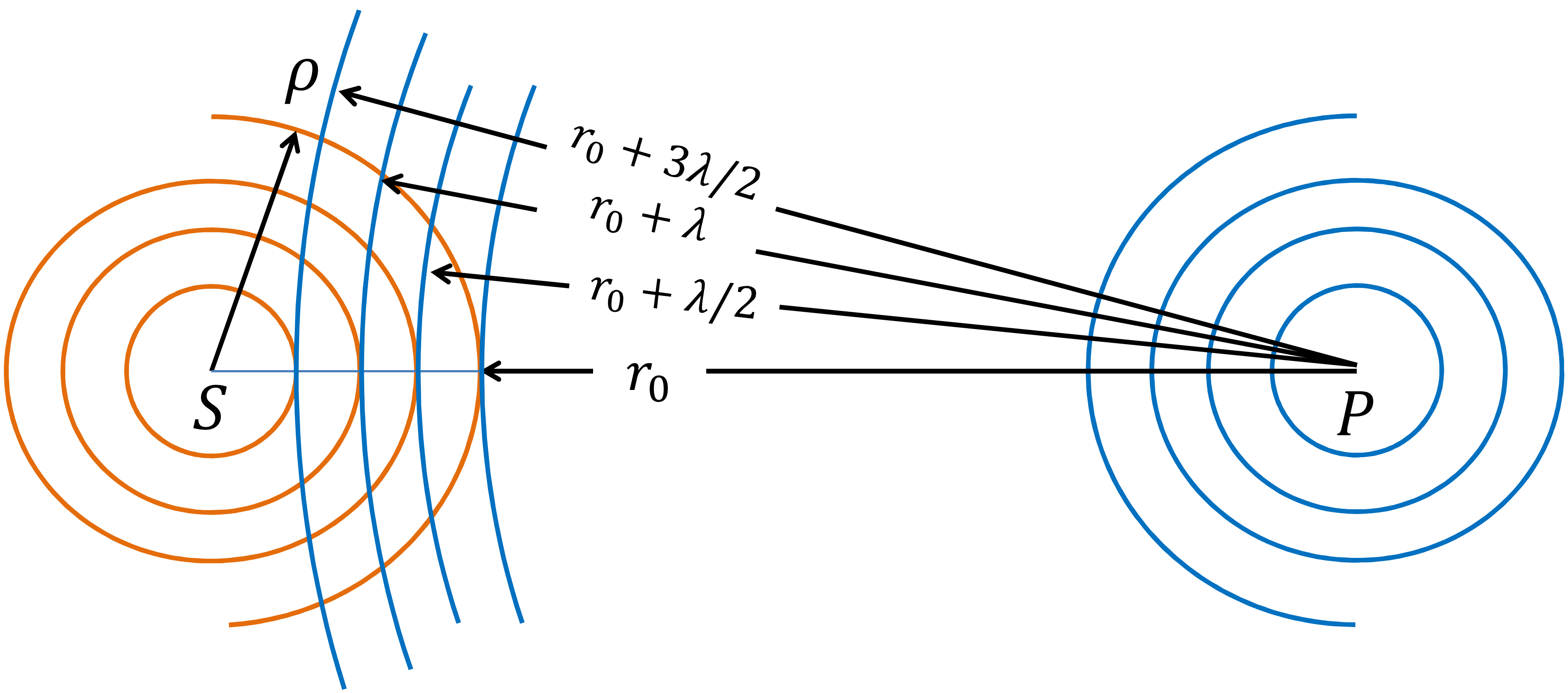
$$E(\rho, t') = \frac{\mathcal{E}_0}{\rho} \cos(\omega t' - k\rho)$$



- These spherical waves expand outward from S

Propagation of Spherical Waves

- Consider a series of concentric spheres around another point P with radii $r_0, r_0 + \lambda/2, r_0 + \lambda, \cdots$



Propagation of Spherical Waves

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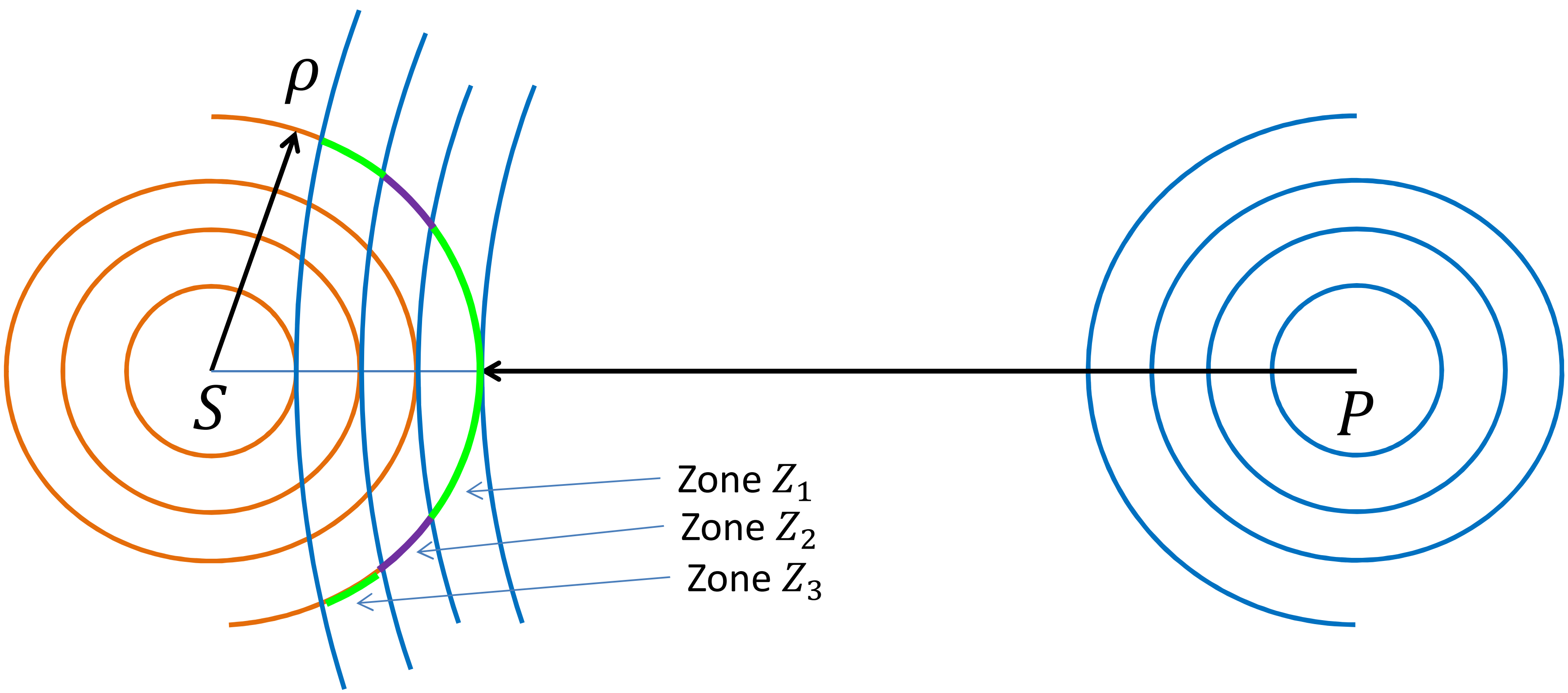
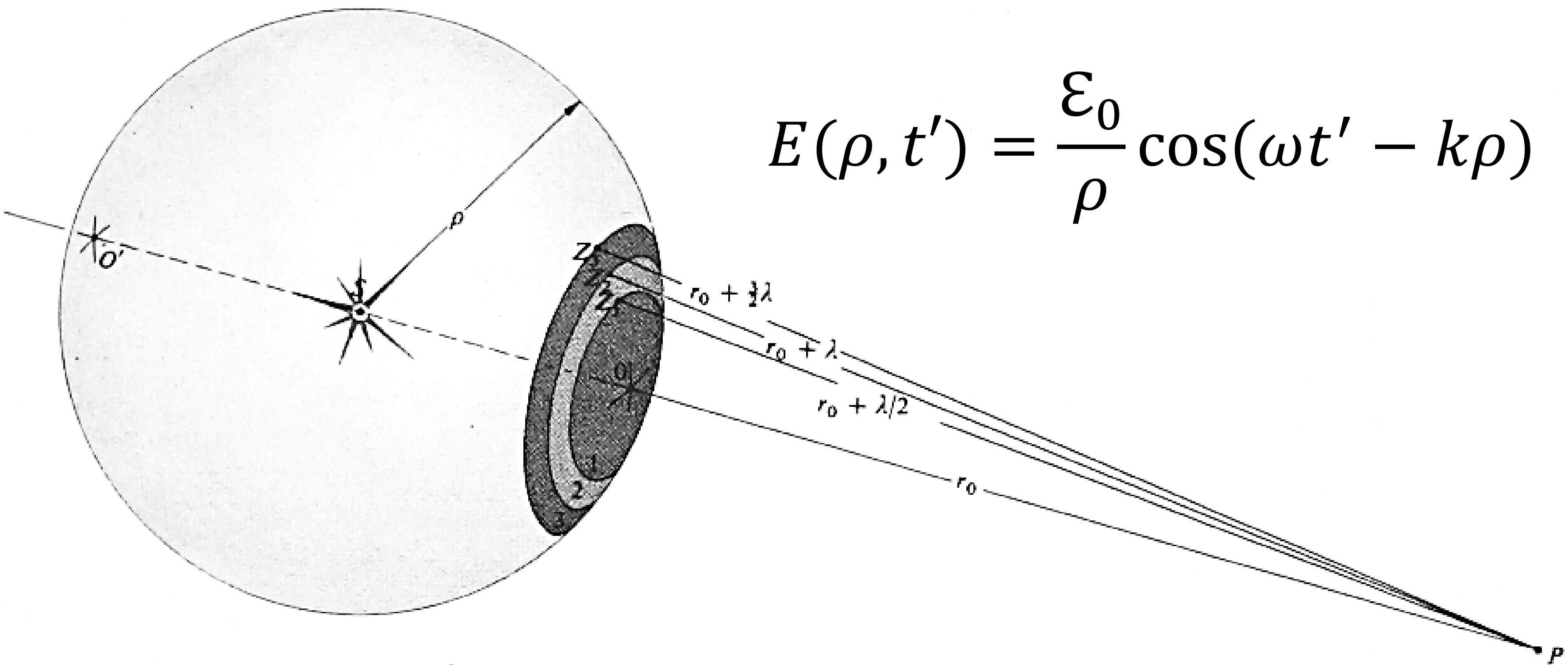


Fresnel Zones



$$E(\rho, t') = \frac{\mathcal{E}_0}{\rho}\cos(\omega t' - k\rho)$$

- Source strength per unit area in any zone is

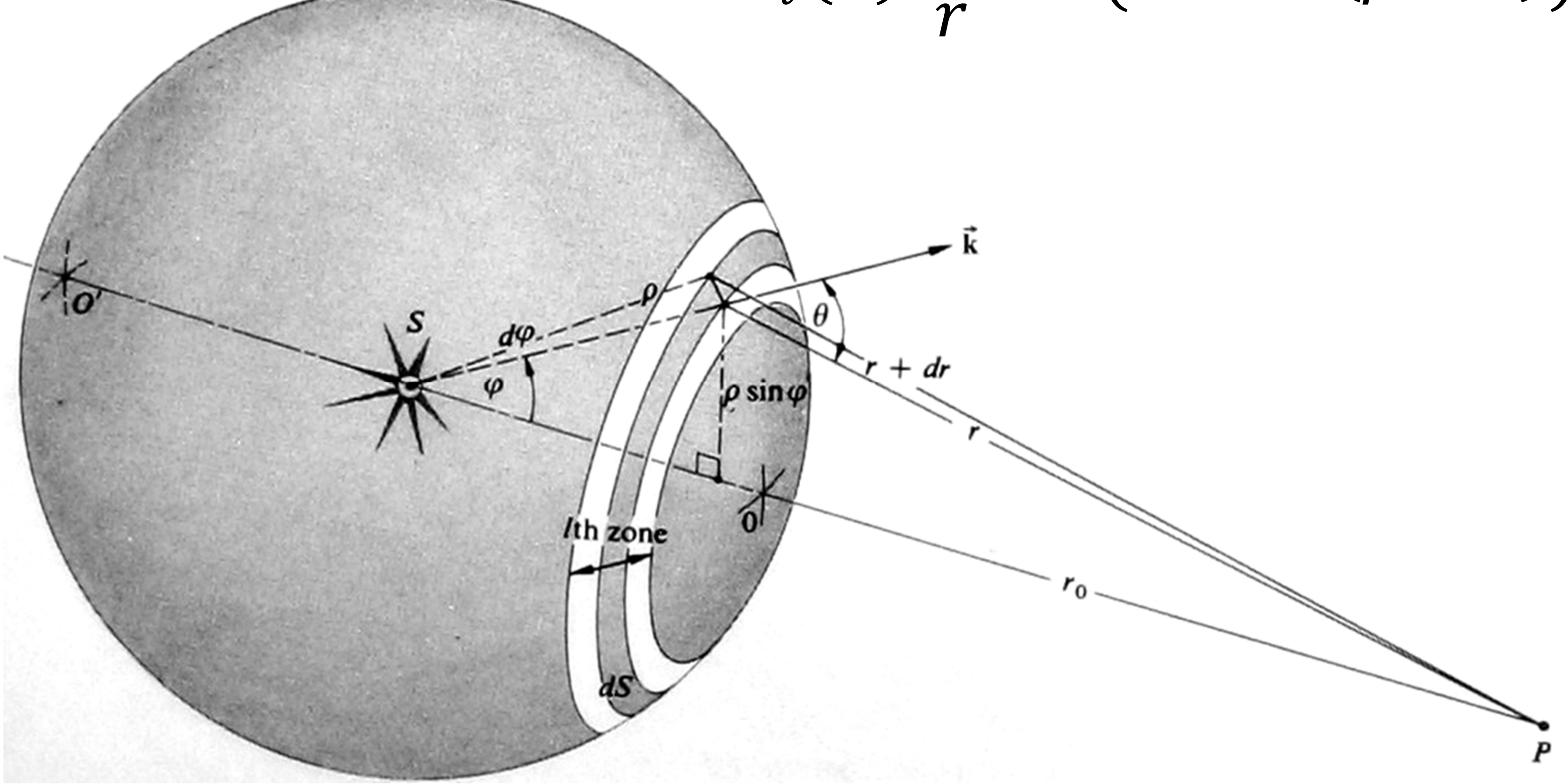
$$\mathcal{E}_A \propto \frac{\mathcal{E}_0}{\rho}$$

- Curvature of the surface in each zone is small

Fresnel Zones

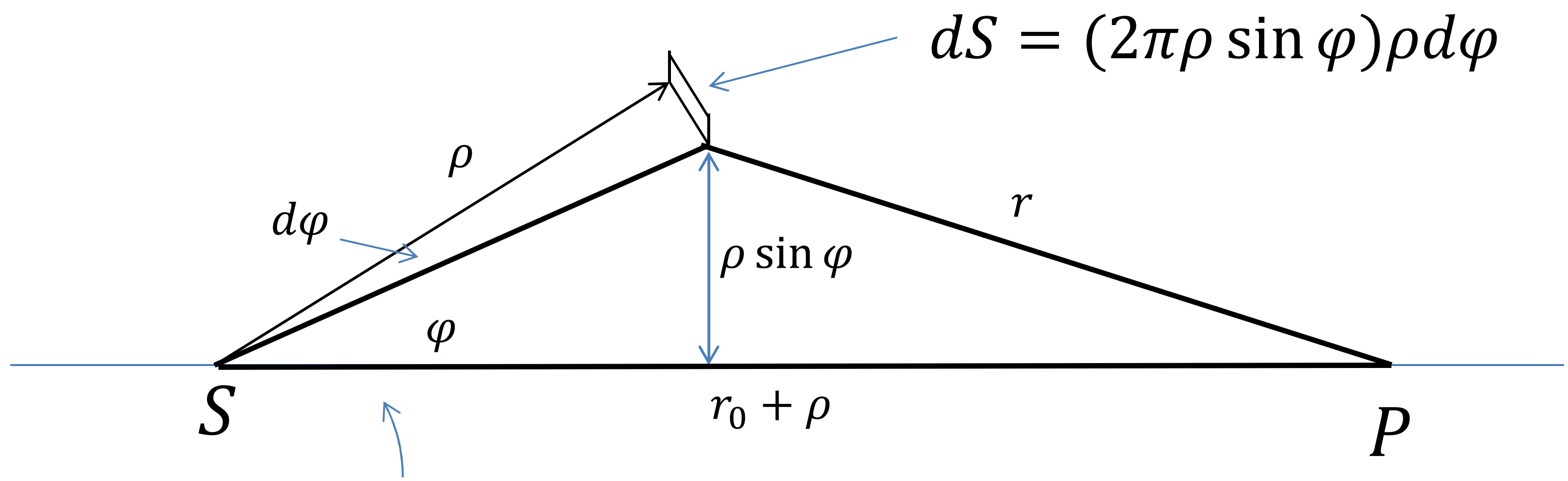
Electric field at point P due to secondary waves in zone ℓ:

$$dE = K_{\ell}(\theta)\frac{\mathcal{E}_A}{r}\cos\big(\omega t - k(\rho + r)\big)\,dS$$



Fresnel Zones

- How can we describe the element of area dS?



Law of cosines: $r^2 = \rho^2 + (r_0 + \rho)^2 - 2\rho(r_0 + \rho)\cos\varphi$

$$2rdr = 2\rho(r_0 + \rho)\sin\varphi\, d\varphi$$

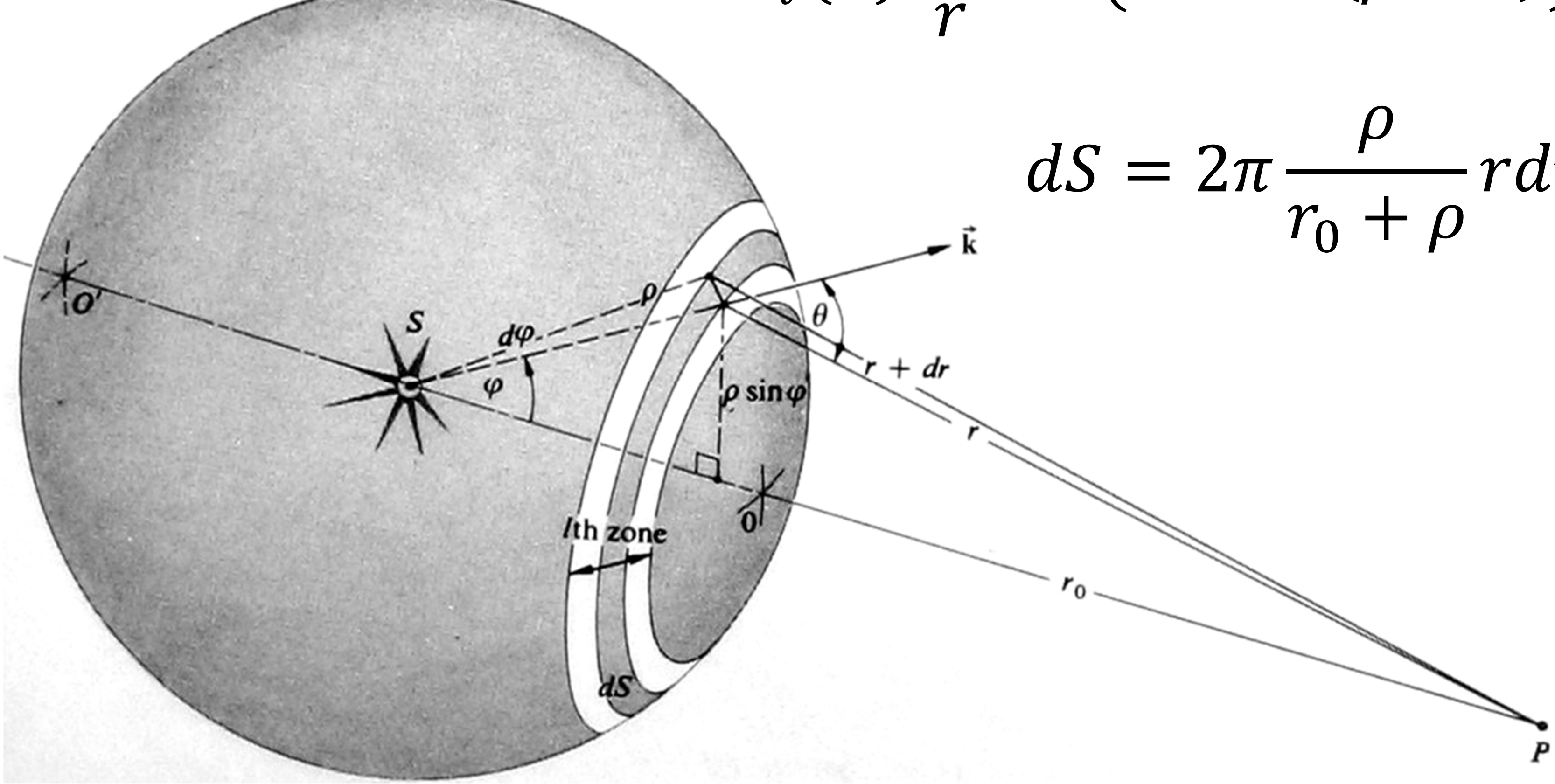
$$\rho\sin\varphi\, d\varphi = \frac{rdr}{r_0 + \rho}$$

Fresnel Zones

Electric field at point P due to secondary waves in zone ℓ:

$$dE = K_\ell(\theta)\frac{\mathcal{E}_A}{r}\cos\big(\omega t - k(\rho + r)\big)\,dS$$

$$dS = 2\pi\frac{\rho}{r_0+\rho}r dr$$



Fresnel Zones

- Total electric field due to wavelets emitted in zone ℓ:

$$E_\ell = 2\pi K_\ell(\theta)\frac{\mathcal{E}_A\rho}{\rho + r_0}\int_{r_{\ell-1}}^{r_\ell}\cos\big(\omega t - k(\rho + r)\big)dr$$

$$= -\frac{2\pi}{k}K_\ell(\theta)\frac{\mathcal{E}_A\rho}{\rho + r_0}\big[\sin\big(\omega t - k(\rho + r)\big)\big]_{r_{\ell-1}}^{r_\ell}$$

- But $r_\ell = r_0 + \ell\lambda/2$ and $r_{\ell-1} = r_0 + (\ell - 1)\lambda/2$ and $k\ell\lambda/2 = \ell\pi$, so

$$E_\ell = (-1)^{\ell+1}K_\ell(\theta)\frac{\mathcal{E}_A\rho\lambda}{\rho + r_0}\sin\big(\omega t - k(\rho + r_0)\big)$$

- Even ℓ: $E_\ell < 0$, odd ℓ: $E_\ell > 0$

Fresnel Zones

- The total electric field at point P is the sum of all electric fields from each zone:

$$E = E_1 + E_2 + E_3 + \cdots + E_m$$
$$= |E_1| - |E_2| + |E_3| - |E_4| + \cdots \pm |E_m|$$

- Most of the adjacent zones cancel:

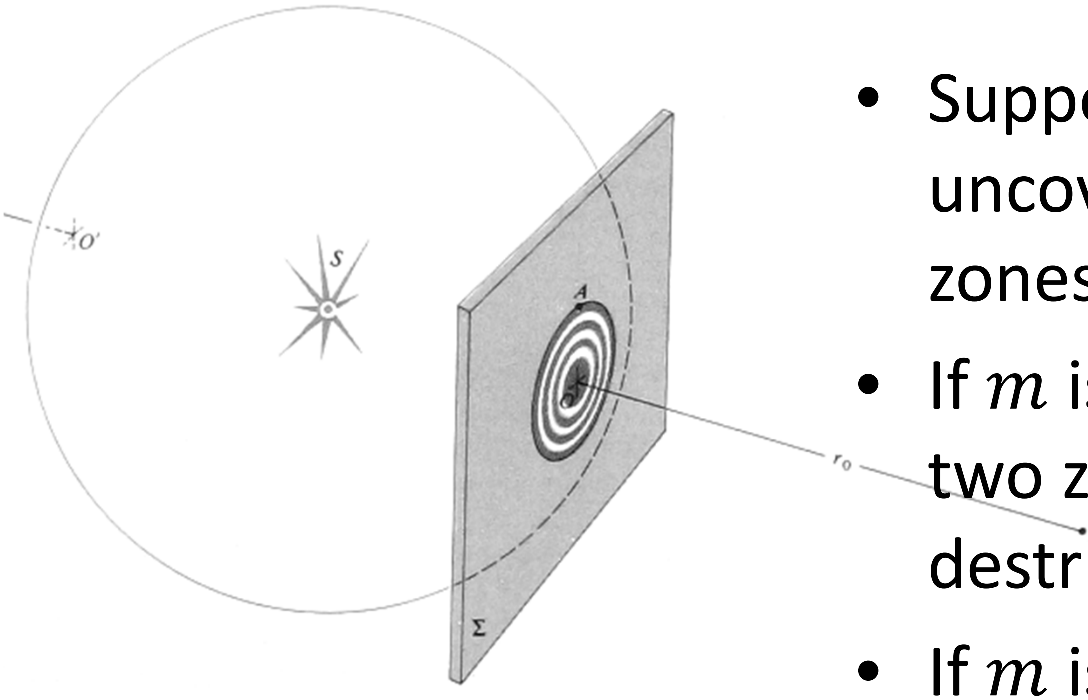
$$E = \frac{|E_1|}{2} + \left(\frac{|E_1|}{2} - |E_2| + \frac{|E_3|}{2}\right) + \cdots \pm \frac{|E_m|}{2}$$

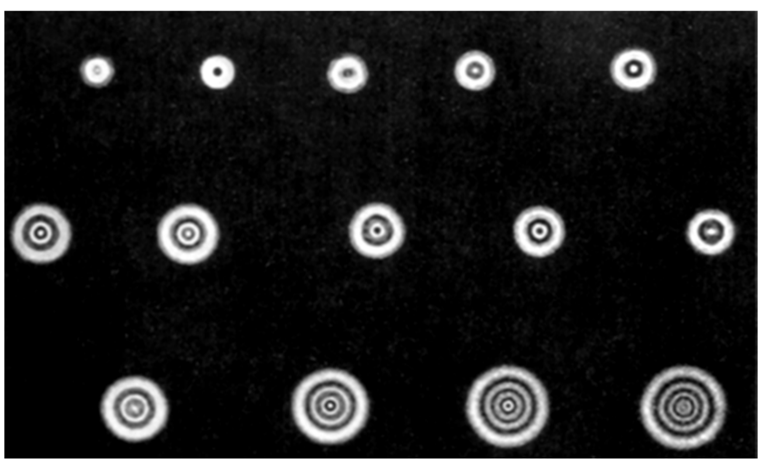
- Two possibilities:

$$E \approx \frac{|E_1|}{2} + \frac{|E_m|}{2} \quad \text{or} \quad E \approx \frac{|E_1|}{2} - \frac{|E_m|}{2}$$

- Fresnel conjectured that $|E_m| \to 0$ so $E \approx |E_1|/2$

Circular Aperture: Fresnel Diffraction

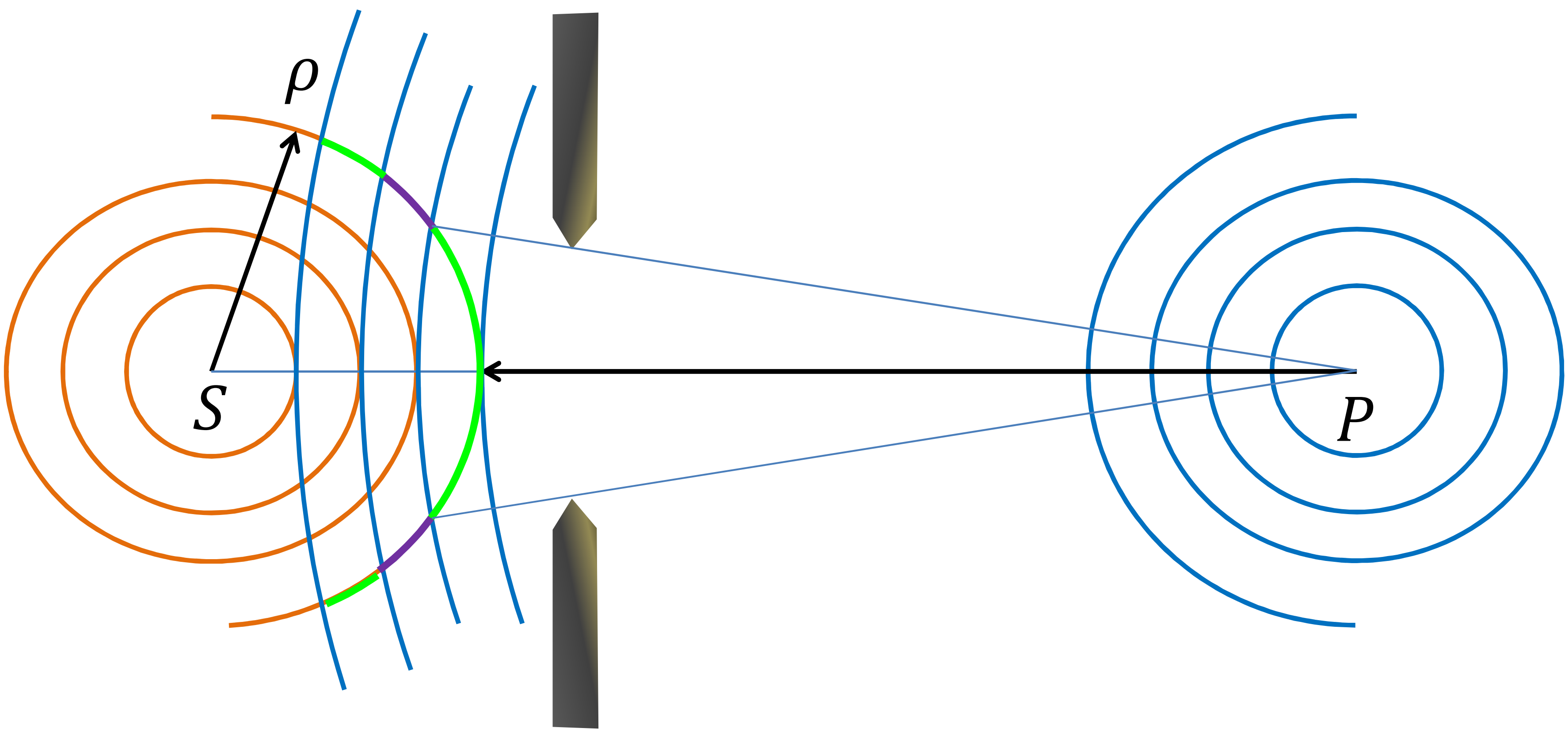




- Suppose a circular aperture uncovers only the first m zones.
- If m is even, then the first two zones interfere destructively: $E = 0$
- If m is odd, then all but the first one cancel each other: $E = |E_1|$
- What about points off the central axis?

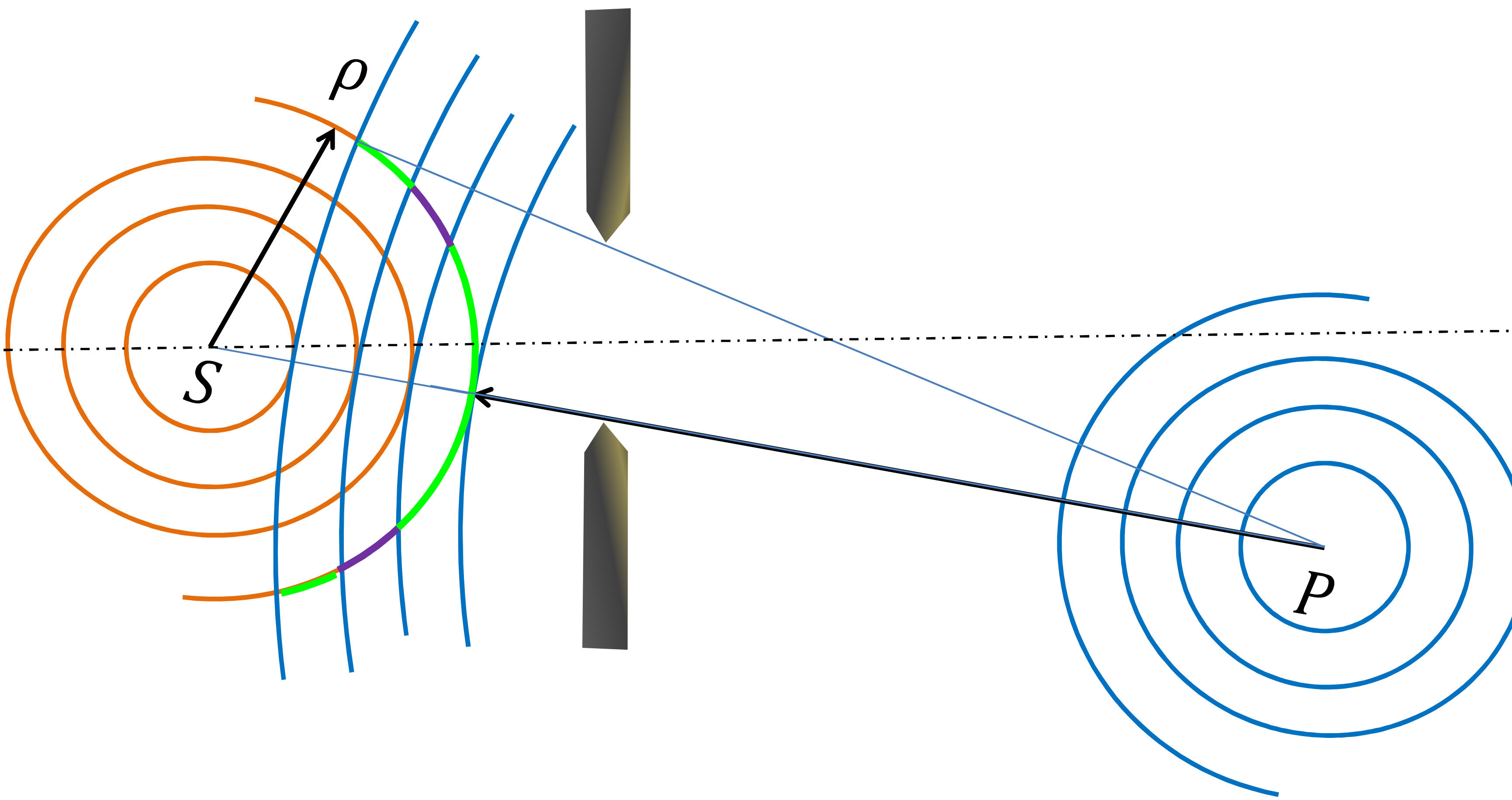
Circular Aperture: Fresnel Diffraction

- A point on the central axis:

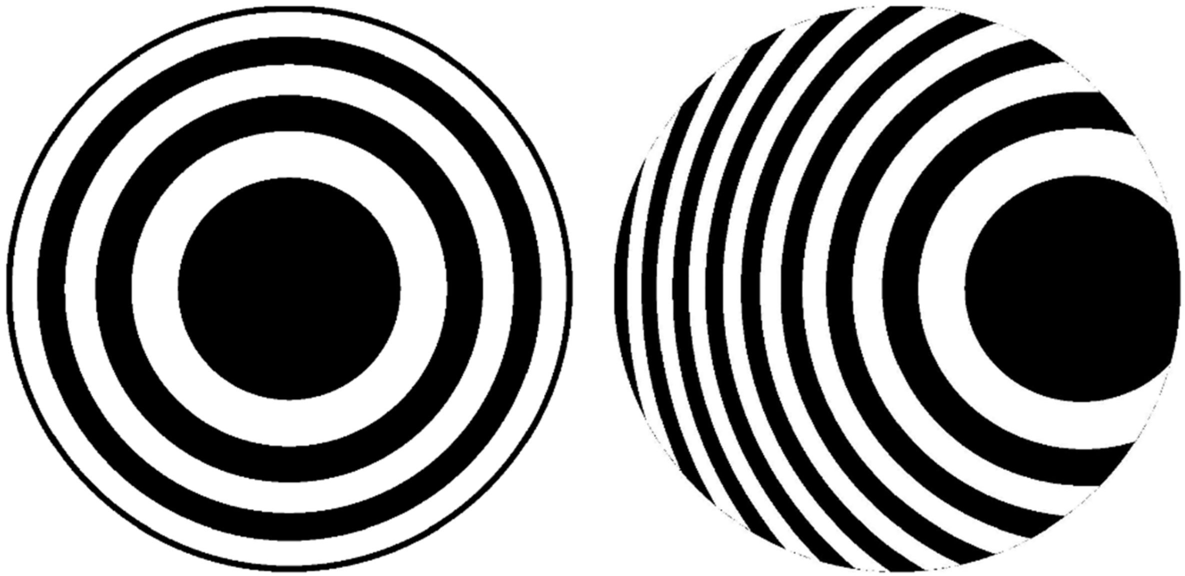


Circular Aperture: Fresnel Diffraction

- A point that is not on the central axis:

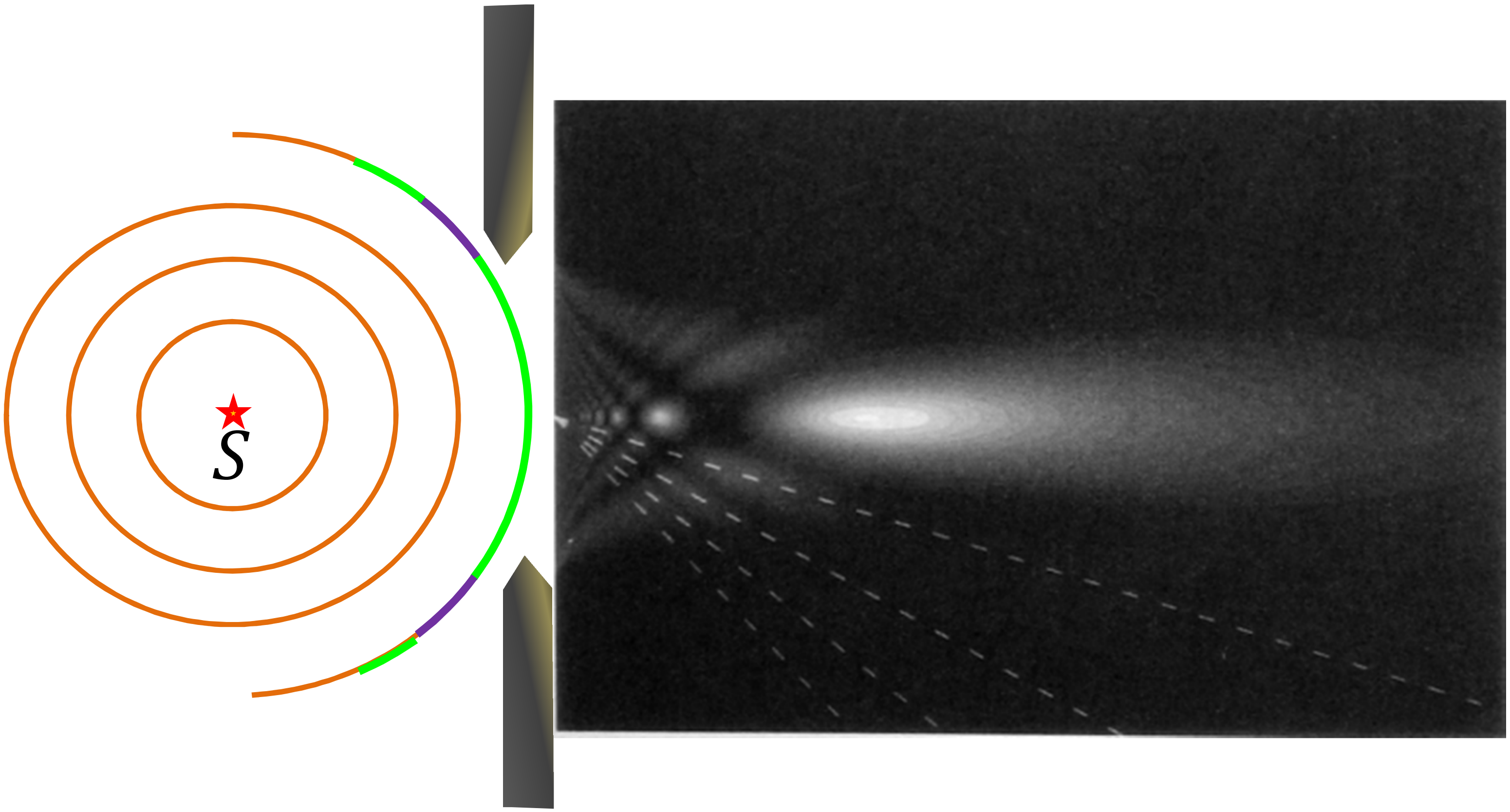


Circular Aperture: Fresnel Diffraction



- There will also be light and dark fringes off the central axis

Circular Aperture: Fresnel Diffraction

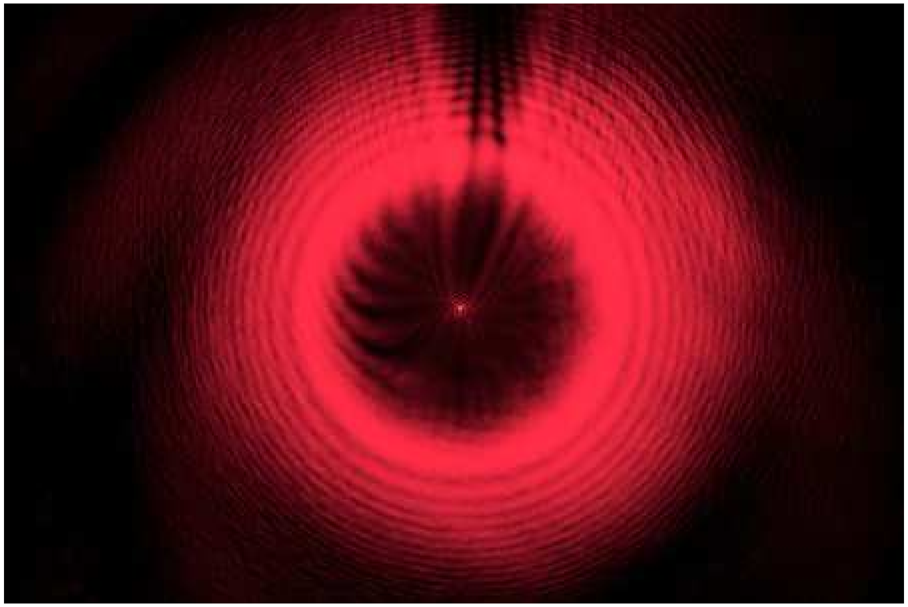


Circular Obstacle: Fresnel Diffraction

- A circular obstacle will remove the middle zones, but the remaining zones can interfere constructively and destructively
$$E = |E_{\ell+1}| - |E_{\ell+2}| + |E_{\ell+3}| \dots \pm |E_m|$$
- As in the case of the unobstructed wave, only the first unobstructed zone contributes:
$$E \approx \frac{|E_{\ell+1}|}{2}$$
- There should be a bright spot on the central axis

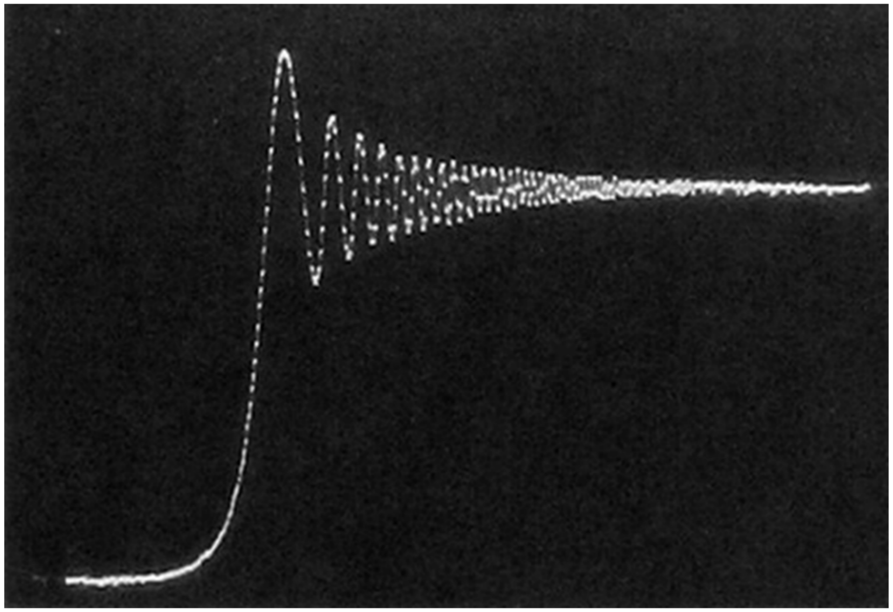
Poisson Bright Spot

- Poisson thought this result seemed absurd and dismissed Fresnel’s paper
- Arago checked and found the bright spot:



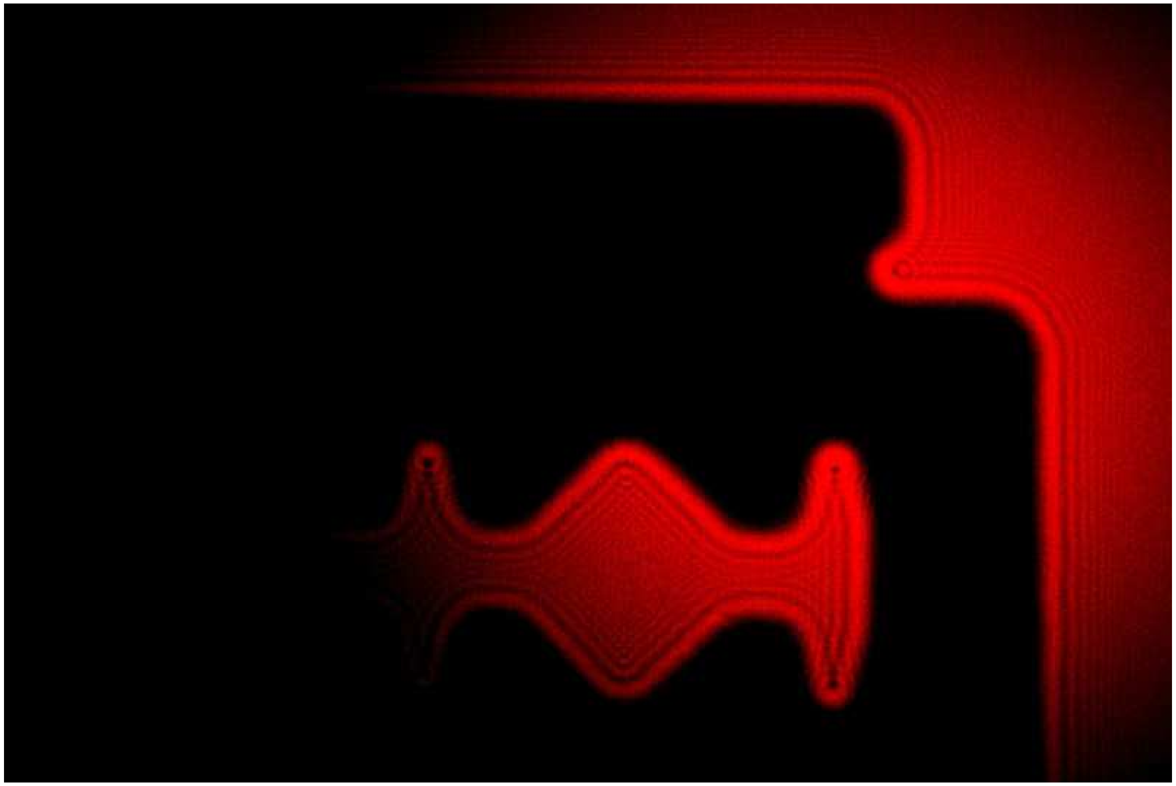
Diffraction around a 1/8” ball bearing

Diffraction from an Edge



- The edge does not form a distinct shadow

Diffraction from an Edge

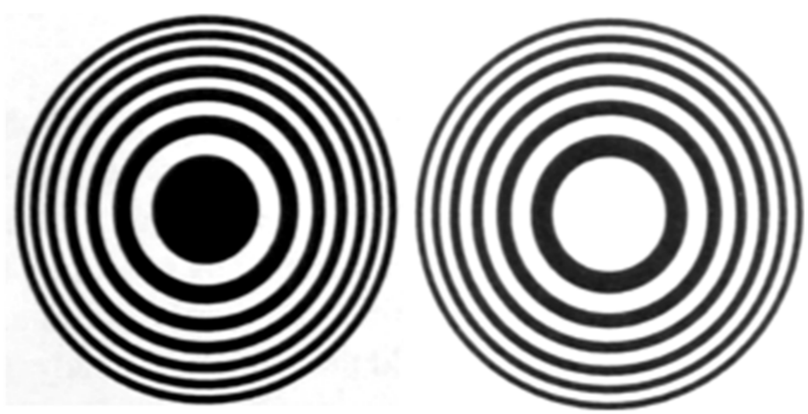


Fresnel Zone Plate

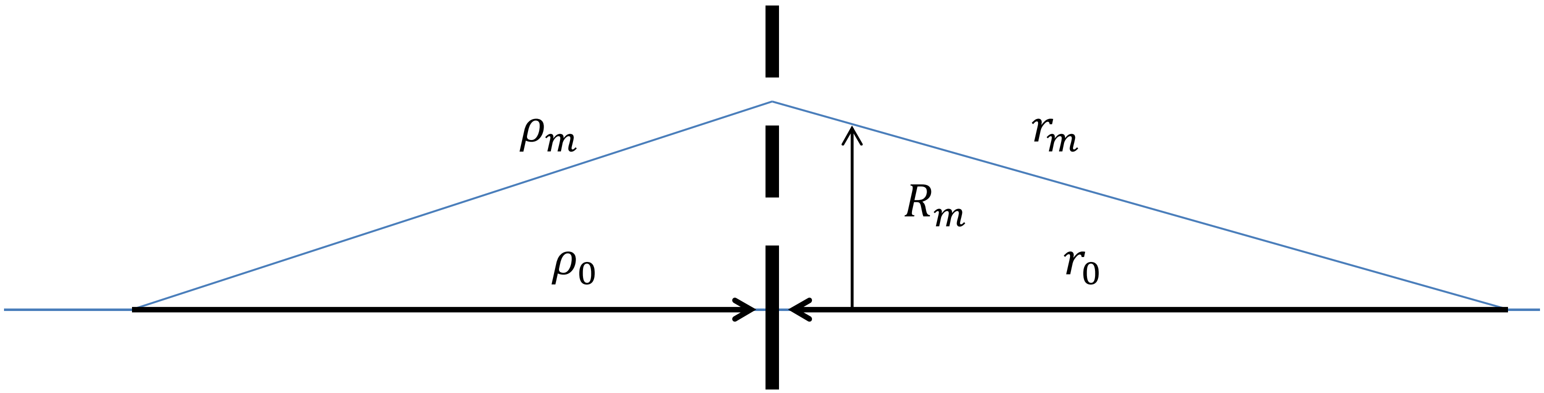
- Suppose we obscure only the even-numbered zones

$$E = |E_1| + |E_2| + |E_5| + \cdots + |E_m|$$

- The electric field at the origin is $2m$ times that of the unobstructed light
- What radii do we need to make some annular rings that block only the even-numbered zones?



Fresnel Zone Plate



$$(\rho_m + r_m) - (\rho_0 - r_0) = \frac{m\lambda}{2}$$

$$\rho_m = \sqrt{\rho_0^2 + R_m^2} \approx \rho_0 + \frac{R_m^2}{2\rho_0}$$

$$r_m = \sqrt{r_0^2 + R_m^2} \approx r_0 + \frac{R_m^2}{2r_0}$$

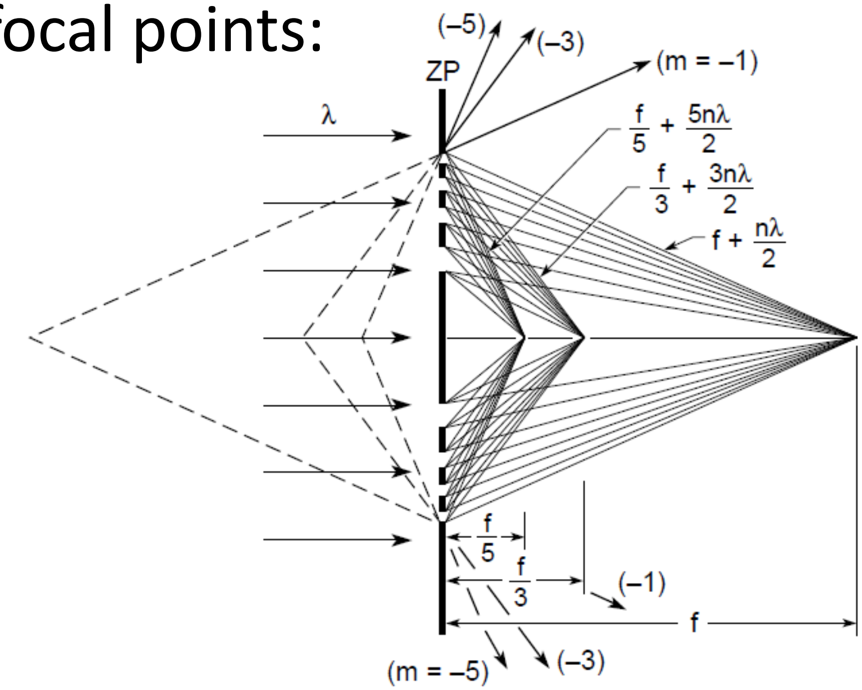
$$\frac{1}{\rho_0} + \frac{1}{r_0} \approx \frac{m\lambda}{R_m^2} = \frac{1}{f}$$

This looks like the lens equation…

Fresnel Zone Plates

- There are also higher-order focal points:

$$f_m = \frac{1}{m} f_1$$



- Not an ideal lens
  – Works only for one wavelength (large chromatic aberration)
- But applicable to a wide range of wavelengths
  – Does not rely on weird atomic properties of transparent materials

Fresnel Zone Plate

$$R_m \approx \sqrt{mf\lambda}$$

- For green light, $\lambda = 500\ nm$
- Suppose $\rho_0 = r_0 = 10\ cm$
  - Then $R_1 = 0.223\ mm$, $R_2 = 0.316\ mm$, etc…
- But this also works for x-rays: $\lambda \sim 0.1\ nm$
  - Then $R_1 = 3.16\ \mu m$, $R_2 = 4.47\ \mu m$
- Challenges: very small spacing, but needs to be thick enough to absorb x-rays.