Assignment #6

1. The angle between the electric field and the axis of the first polarizer is 30° so the component that is passed through the first polarizer is

$$E_1 = E_0 \cos 30^\circ = (0.866) E_0 .$$

The second polarizer is oriented with its polarization axis at 50° with respect to the first polarizer. The component of the electric field that passes the second polarizer is

$$E_2 = E_1 \cos 50^\circ = (0.643) E_1$$
$$= (0.557) E_0 .$$

Irradiance is proportional to the square of the electric field so the fraction of the irradiance that emerges is

$$f = E_2^2 / E_0^2 = (0.557)^2 = 0.310$$

or 31%.

2. Light is incident on an air-glass interface at $40°$.

The reflection coefficient for the component which has its $\vec{E}$-field perpendicular to the plane of incidence is

$$\left(\frac{E_r}{E_i}\right)_\perp = -\frac{\sin(\theta_i - \theta_t)}{\sin(\theta_i + \theta_t)}$$

and the reflection coefficient for the component which has its $\vec{E}$-field parallel to the plane of incidence is

$$\left(\frac{E_r}{E_i}\right)_\parallel = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)}$$

The angle of transmission is obtained from Snell's law:

$$n_i \sin\theta_i = n_t \sin\theta_t$$

$$\text{so} \quad \sin\theta_t = \frac{n_i \sin\theta_i}{n_t} = \frac{\sin 40°}{1.5}$$

$$= 0.4285$$

$$\Rightarrow \theta_t = 25.37°$$

Thus, $\left(\frac{E_r}{E_i}\right)_\perp = -\frac{\sin(40° - 25.37°)}{\sin(40° + 25.37°)} = -\frac{\sin(14.63°)}{\sin(65.37°)}$

$$= -0.278$$

$$\left(\frac{E_r}{E_i}\right)_\parallel = \frac{\tan(14.63°)}{\tan(65.37°)} = 0.120$$

The irradiance of the two components is

$$I_\perp \propto (-0.278)^2 = 0.0773$$

$$I_\parallel \propto (0.120)^2 = 0.0144$$

The polarized irradiance is

$$I_p \propto 0.0773 - 0.0144 = 0.0629$$

and the unpolarized irradiance is

$$I_n \propto 2(0.0144) = 0.0288 .$$

The degree of polarization is then

$$V = \frac{0.0629}{0.0629 + 0.0288} = 0.686$$

or 68.6%

Alternatively, we can compute the degree of polarization using

$$V = \frac{I_\perp - I_\parallel}{I_\perp + I_\parallel} = \frac{0.0773 - 0.0144}{0.0773 + 0.0144}$$

$$= 0.686 \text{ or } 68.6\%$$

3. The Mueller matrix for a linear polarizer with its transmission axis at 45° is

$$M = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The Stokes parameters for unpolarized light are

$$S = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

After passing through the polarizer, the Stokes parameters are

$$S' = MS = \frac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

which corresponds to light that is polarized at 45°.

The relative irradiance is 1/2.

The degree of polarization is

$$V = \frac{\sqrt{S_1^2 + S_2^2 + S_3^2}}{S_0} = \frac{\sqrt{1/4}}{1/2} = 1.$$

4. A quarter wave plate introduces a phase difference of $\pi/2$ between the components aligned parallel or perpendicular to the fast axis.

Horizontal polarized light incident on a quarter wave plate with its fast axis at 45° will result in light emerging with electric field components

$$\vec{E}_f = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) E_o \cos(kz - \omega t)$$

$$\vec{E}_s = \frac{1}{\sqrt{2}}(\hat{\imath} - \hat{\jmath}) E_o \cos(kz - \omega t + \pi/2)$$

The resulting electric field is

$$\vec{E} = \frac{\hat{\imath} E_o}{\sqrt{2}}\Big(\cos(kz - \omega t) + \cos(kz - \omega t + \pi/2)\Big)$$

$$+ \frac{\hat{\jmath} E_o}{\sqrt{2}}\Big(\cos(kz - \omega t) - \cos(kz - \omega t + \pi/2)\Big)$$

Using the identities,

$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\cos A - \cos B = 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(B-A)$$

the electric field can be written

$$\vec{E} = \frac{2E_o\hat{\imath}}{\sqrt{2}}\Big(\cos(kz - \omega t + \pi/4)\cos(\pi/4)\Big)$$

$$+ \frac{2E_o\hat{\jmath}}{\sqrt{2}}\Big(\sin(kz - \omega t + \pi/4)\sin(\pi/4)\Big)$$

$$= E_o\Big(\hat{\imath}\cos(kz - \omega t + \pi/4) + \hat{\jmath}\sin(kz - \omega t + \pi/4)\Big)$$

which describes right-circular polarization.

(a) Horizontal polarized light is described by the Stokes parameters

$$S = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

After passing through the quarter-wave plate, with Mueller matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

the stokes parameters are

$$S' = MS = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

which corresponds to right circular polarization.

(b) Light that is linearly polarized at 45° has Stokes parameters

$$S = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

After emerging from the quarter wave plate,

$$S' = MS = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

so it is still linearly polarized at 45°.

(c) This question was addressed in the analysis of part (a). The result is right-circular polarized light.

5. The Jones matrix for the liquid cell is

$$M = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{3} & -1+\sqrt{3} \\ 1-\sqrt{3} & 1+\sqrt{3} \end{pmatrix}$$

(a) Horizontal polarized light has a Jones vector

$$J = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

The light emerging from the cell has a Jones vector

$$J' = MJ = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{3} \\ 1-\sqrt{3} \end{pmatrix}$$

There is no phase difference introduced so the light is still linearly polarized. The angle of the plane of polarization is given by

$$\tan\theta = \frac{E_y}{E_x} = \frac{1-\sqrt{3}}{1+\sqrt{3}}$$

$$\Rightarrow \quad \theta = -15^\circ$$

(b) If the incident light was vertically polarized then

$$J = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad J' = MJ = \frac{1}{2\sqrt{2}} \begin{pmatrix} -1+\sqrt{3} \\ 1+\sqrt{3} \end{pmatrix}$$

$$\tan\theta = \frac{1+\sqrt{3}}{-1+\sqrt{3}} \Rightarrow \theta = 75^\circ$$

(c) The rotation angle is $\alpha = -15^\circ$.