1. When the solution to the differential equation

$$m\ddot{x} + b\dot{x} + kx = 0$$

is written $x(t) = Ae^{-\gamma t/2}\cos\omega t$ then we are explicitly considering underdamped oscillations.

The total energy is

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

where the time derivative is

$$\dot{x} = -\frac{A\gamma}{2}e^{-\gamma t/2}\cos\omega t - A\omega e^{-\gamma t/2}\sin\omega t .$$

The kinetic energy can then be written

$$T = \frac{1}{2}mA^2\left(\frac{\gamma^2}{4}e^{-\gamma t}\cos^2\omega t + \omega\gamma e^{-\gamma t}\sin\omega t\cos\omega t + \omega^2 e^{-\gamma t}\sin^2\omega t\right)$$

and the potential energy is

$$V = \frac{1}{2}kA^2 e^{-\gamma t}\cos^2\omega t .$$

The interpretation of the expression for the total energy is not immediately transparent from these expressions as written. However, in the limit where $\gamma \ll \omega$ we have

$$\omega = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}} \approx \omega_0 = \sqrt{\frac{k}{m}}$$

In this limit then,

$$E \approx \frac{1}{2} m A^2 \omega_0^2 e^{-\gamma t} \sin^2 \omega t + \frac{1}{2} k A^2 e^{-\gamma t} \cos^2 \omega t$$

$$= \frac{1}{2} A^2 k e^{-\gamma t} (\sin^2 \omega t + \cos^2 \omega t)$$

$$= \frac{1}{2} A^2 k e^{-\gamma t}$$

$$= E_0 e^{-\gamma t}$$

where $E_0 \equiv \frac{1}{2} k A^2$ is the initial energy of the oscillator.

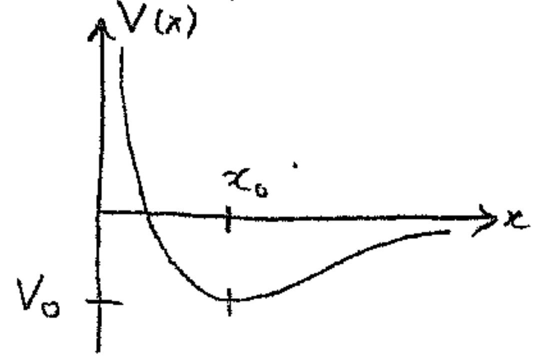
The point of this exercise is to see that the energy decreases as $e^{-\gamma t}$.

2. Consider the potential

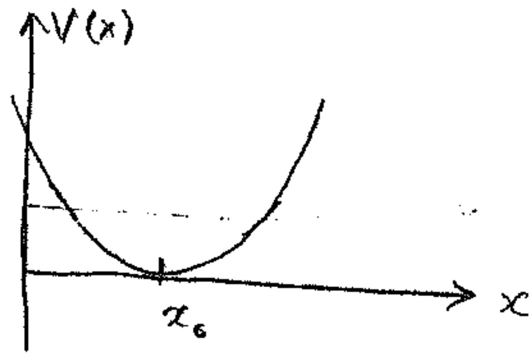
$$V(x) = \frac{b}{2x^2} - \frac{a}{x}$$

where we assume $x > 0$.

(a) The graph of the potential looks like this:



which can be compared with the potential for a spring with equilibrium position x_0:



Both curves are concave upwards and have exactly one minimum located at $x = x_0 > 0$

(b). The force is determined from the potential function:

$$F = -\frac{dV}{dx} = -\frac{b}{x^3} + \frac{a}{x^2}$$

and when $F = 0$ at $x = x_0$,

$$\frac{a}{x_0^2} = \frac{b}{x_0^3} \Rightarrow x_0 = \frac{b}{a}.$$

(c) If we write $x' = x - x_0$ then $x = x' + x_0$ and the potential can be written

$$V(x') = \frac{b}{2(x'+x_0)^2} - \frac{a}{x'+x_0}$$

This function will have its minimum located at $x' = 0$.

(d) The Taylor expansion for $V(x')$ can be written

$$V(x') = a_0 + a_1 x' + \frac{a_2}{2}(x')^2 + \mathcal{O}(x'^3)$$

The coefficients are $a_0 = V\Big|_{x'=0} = \frac{b}{2x_0^2} - \frac{a}{x_0}$

$$= \frac{b}{2}\left(\frac{a}{b}\right)^2 - a\left(\frac{a}{b}\right)$$

$$= \frac{1}{2}\frac{a^2}{b} - \frac{a^2}{b}$$

$$= -\frac{1}{2}\frac{a^2}{b}$$

The coefficient a_1 must vanish because

$$F = -\frac{dV}{dx'} = 0 \quad \text{at } x' = 0.$$

The coefficient a_2 is calculated from

$$a_2 = \frac{d^2V}{dx'^2}\Big|_{x'=0}$$

$$= \frac{d}{dx}\left(-\frac{b}{(x'-x_0)^3} + \frac{a}{(x'-x_0)^2}\right)\Big|_{x'=0}$$

$$= \frac{3b}{(x'-x_0)^4} - \frac{2a}{(x'-x_0)^3}\Big|_{x'=0}$$

$$= \frac{3b}{x_0^4} - \frac{2a}{x_0^3} = 3b\left(\frac{a}{b}\right)^4 - 2a\left(\frac{a}{b}\right)^3$$

$$= \frac{3a^4}{b^3} - \frac{2a^4}{b^3} = \frac{a^4}{b^3}.$$

Thus, the Taylor series can be expressed

$$V(x') = -\frac{1}{2}\frac{a^2}{b} + \frac{1}{2}\frac{a^4}{b^3}(x')^2$$

(e) The total energy can be written

$$E = T + V$$

$$= \frac{1}{2}m(\dot{x}')^2 + \frac{1}{2}\frac{a^4}{b^3}(x')^2 - \frac{1}{2}\frac{a^2}{b}$$

(f) This can be compared with the total energy of a mass-spring system:

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$ which oscillates

with frequency $\omega_0 = \sqrt{k/m}$. For the given potential, we expect small oscillations to have

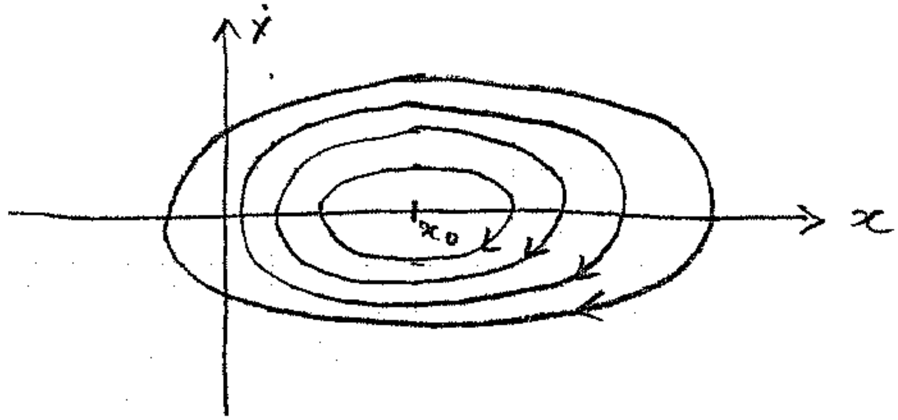
$$\omega_0 = \sqrt{a^4/mb^3}.$$

3a. The phase diagram for a mass-spring system which has a force given by

$$F = -k(x - x_0)$$

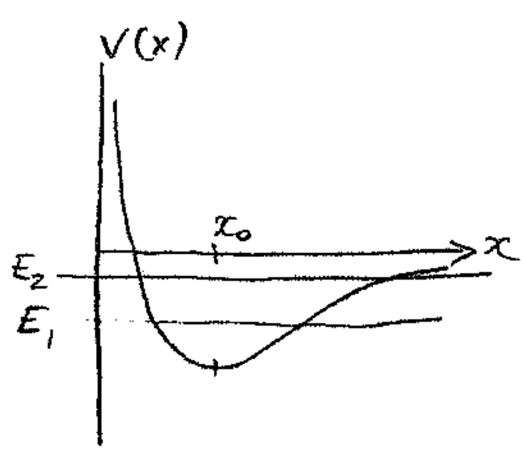
is the same as for the case with $F = -kx$ but with the origin shifted by an offset x_0.

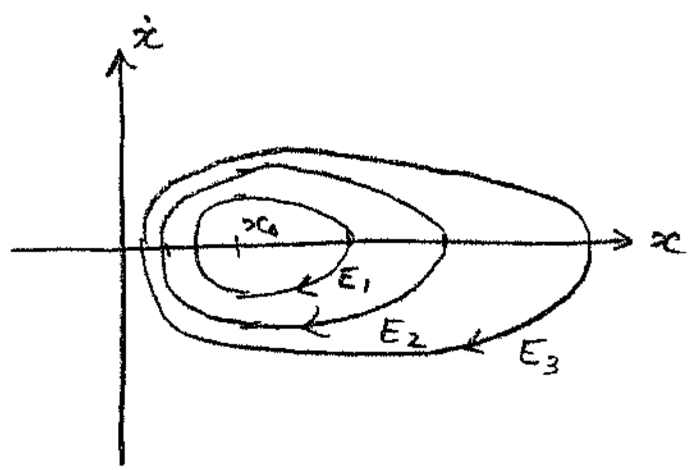
Thus, the phase diagram would look like ellipses of constant E centered at $\dot{x} = 0$, $x = x_0$



b. For the potential $V(x) = \dfrac{b}{2x^2} - \dfrac{a}{x}$,

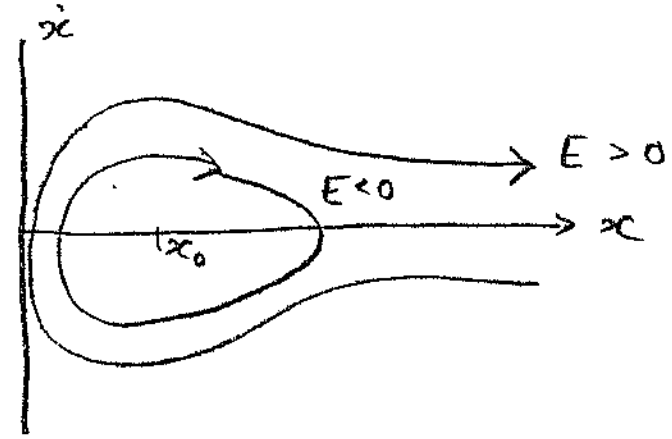
small oscillations about $x = x_0$ will resemble ellipses, but these will be bounded by $x = 0$.





3b. (cont) However, when $E > 0$ the solutions are not periodic. A mass that is incident from $x > 0$ with $\dot{x} < 0$ will approach the origin, but then turn around and never return.

In this case the phase diagram looks like this:



The point of this exercise was to show that we can use the phase diagram to describe many properties of the solutions without explicitly solving the differential equation.