PURDUE UNIVERSITY DEPARTMENT OF PHYSICS

Physics 42200

Waves & Oscillations

Lecture 15 – French, Chapter 6

Spring 2013 Semester

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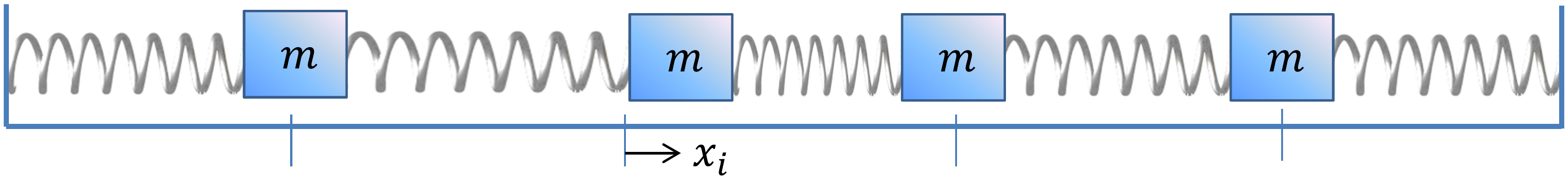
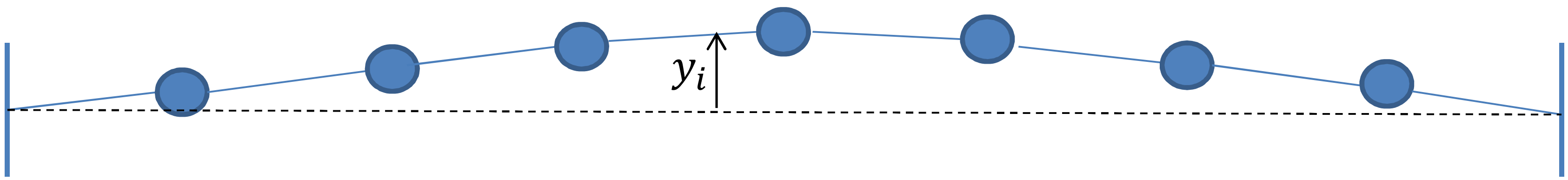
Continuous Systems





$$m\,\ddot{y}_n = F_n = \frac{T}{\ell}\left[(y_{n+1} - y_n) - (y_n - y_{n-1})\right]$$

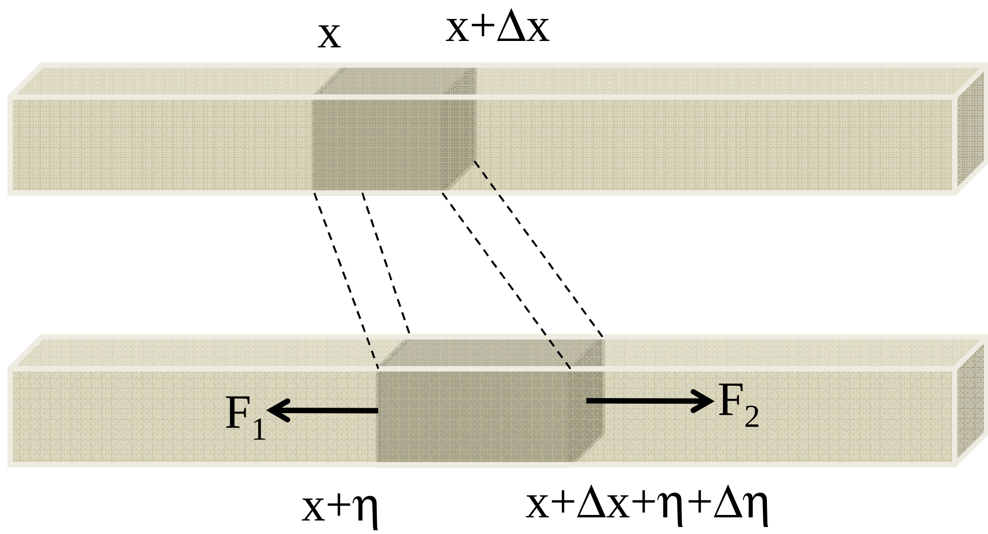
In the limit $m/\ell \rightarrow \mu$ this becomes the wave equation:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

where $v = \sqrt{T/\mu}$.

Other Continuous Systems

- Longitudinal waves in a solid rod:

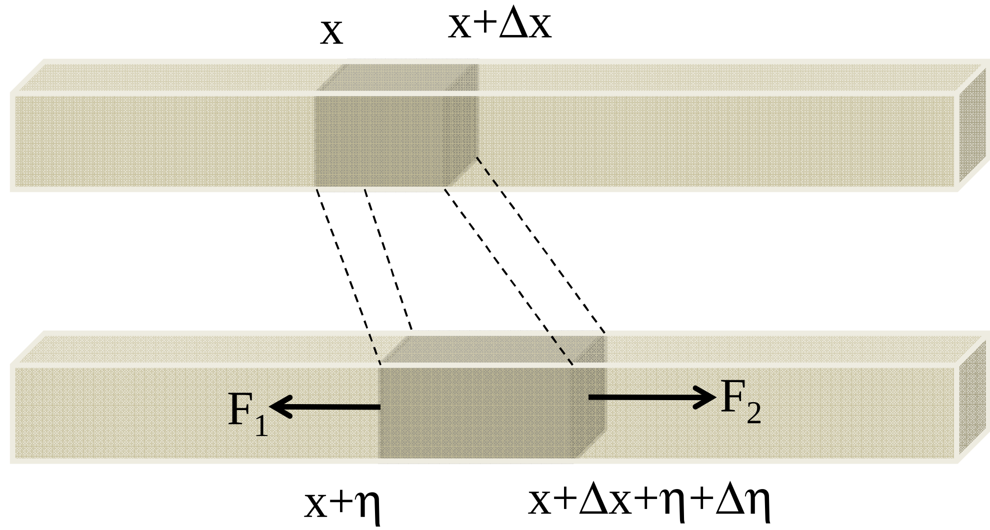


Notation:

- x labels which piece of the rod we are considering, analogous to the index n when counting discrete masses.
- η quantifies how much the element of mass has moved.

- Recall that strain was defined as the fractional increase in length of a small element: $\Delta\eta/\Delta x$
- Stress was defined as $\Delta F/A$
- These were related by $\Delta F/A = Y\ \Delta\eta/\Delta x$

Longitudinal Waves in a Solid Rod



$$\Delta F/A = Y\,\Delta\eta/\Delta x$$

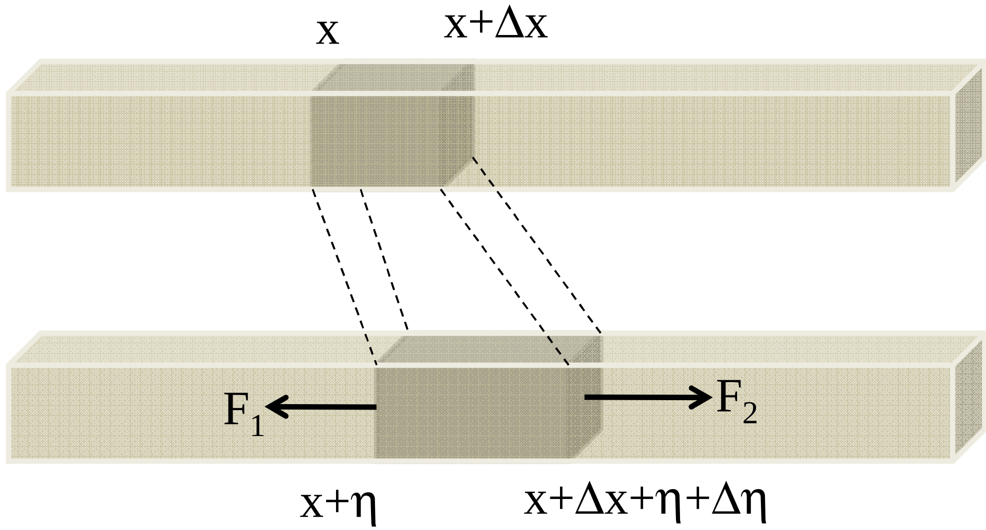
- Force on one side of the element:

$$F_1 = AY\,\Delta\eta/\Delta x = AY\partial\eta/\partial x$$

- Force on the other side of the element:

$$F_2 = F_1 + AY\frac{\partial^2\eta}{dx^2}\Delta x$$

Longitudinal Waves in a Solid Rod



- Newton's law:

$$m\ddot{\eta} = F_2 - F_1$$

$$F_2 - F_1 = AY\frac{\partial^2\eta}{dx^2}\Delta x = \rho A\Delta x\ \frac{\partial^2\eta}{\partial t^2}$$

- Wave equation:

$$\frac{\partial^2\eta}{dx^2} = \frac{\rho}{Y}\frac{\partial^2\eta}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2\eta}{\partial t^2}$$

Longitudinal Normal Modes

- What is the solution for a rod of length L?

$$\frac{\partial^2 \eta}{dx^2} = \frac{1}{v^2}\frac{\partial^2 \eta}{\partial t^2} \qquad v = \sqrt{Y/\rho}$$

- Boundary conditions:
  - Suppose one end is fixed

$$\eta(0) = 0$$

  - No force at the free end of the rod so the stress is zero there. Strain ∝ stress, so the strain is also zero.

$$F = AY\partial\eta/\partial x$$

$$\frac{\partial \eta}{\partial x}_{x=L} = 0$$

- Look for solutions that are of the form

$$\eta(x) = f(x)\cos\omega t$$

Longitudinal Normal Modes

$$\eta(x) = f(x)\cos\omega t$$

- Inspired by the continuous string problem, we let
$$f(x) = A\sin(kx)$$
- Derivatives:
$$\frac{\partial^2\eta}{\partial x^2} = -k^2\eta$$
$$\frac{\partial^2\eta}{\partial t^2} = -\omega^2\eta$$
$$\frac{\partial^2\eta}{dx^2} = \frac{1}{v^2}\frac{\partial^2\eta}{\partial t^2} \Rightarrow k = \frac{\omega}{v}$$

Longitudinal Normal Modes

$$f(x) = A\sin\left(\frac{\omega x}{v}\right)$$

- This automatically satisfies the boundary condition at $x = 0$.
- At $x = L$, $\partial\eta/\partial x = 0$:

$$\frac{\partial\eta}{\partial x}_{x=L} \propto \cos\left(\frac{\omega L}{v}\right) = 0$$

- This means that $\frac{\omega L}{v} = (n - \frac{1}{2})\pi$
- Angular frequencies of normal modes are

$$\omega_n = \frac{\pi}{L}\,(n - \frac{1}{2})\sqrt{Y/\rho}$$

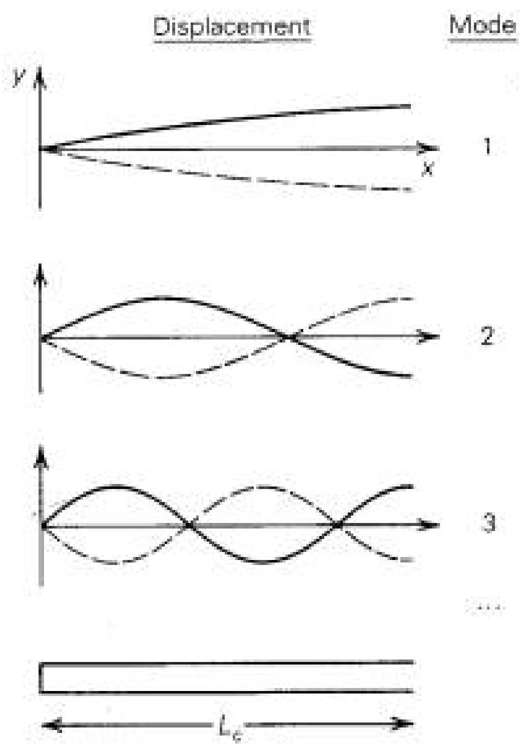
- Frequencies of normal modes are

$$\nu_n = \frac{n - 1/2}{2L}\sqrt{Y/\rho}$$

Longitudinal Normal Modes

- Frequencies of normal modes are

$$\nu_n = \frac{n - 1/2}{2L}\sqrt{Y/\rho}$$



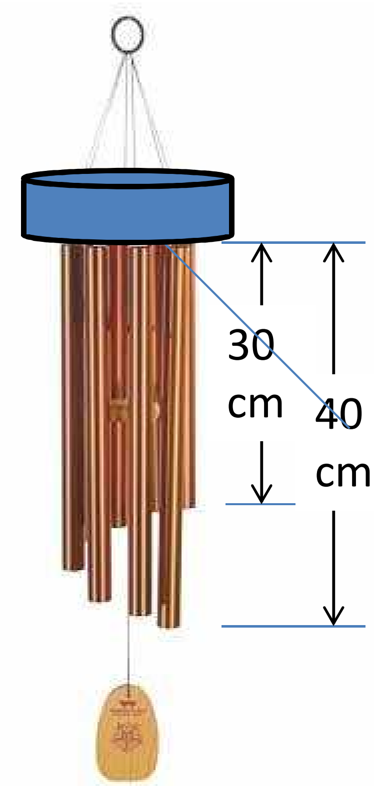
Lowest possible frequency:

$$\nu_1 = \frac{1}{4L}\sqrt{\frac{Y}{\rho}}$$

Frequencies of Metal Chimes

- Suppose a set of chimes were made of copper rods, with lengths between 30 and 40 cm, rigidly fixed at one end.
- What frequencies should we expect if

$$Y = 117 \times 10^9\ \mathrm{N} \cdot m^{-2}$$
$$\rho = 8.96 \times 10^3\ \mathrm{kg} \cdot m^{-3}$$



$$\nu_1 = \frac{1}{4L}\sqrt{\frac{117 \times 10^9\ \mathrm{N} \cdot m^{-2}}{8.96 \times 10^3\ \mathrm{kg} \cdot m^{-3}}}$$
$$= 2260\ - 3010\ Hz$$

(highest octave on a piano)

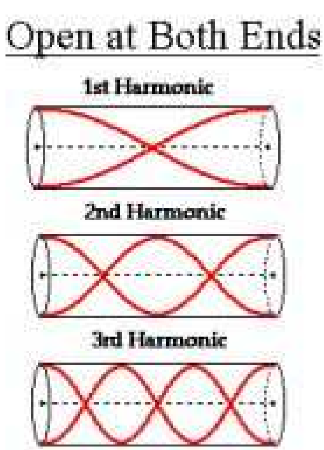
Frequencies of Metal Chimes

- If the metal rods were not fixed at one end then the boundary conditions at both ends would be:

$$\frac{\partial \eta}{\partial x} = 0$$

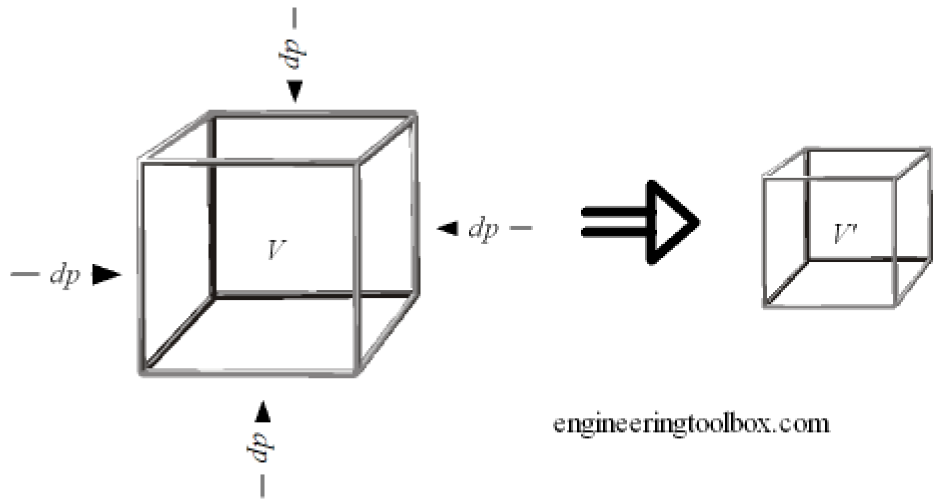
- Allowed frequencies of normal modes:

$$\nu_n = \frac{n}{2L}\sqrt{Y/\rho}$$



Harmonic	Wavelength λ	Frequency f
1^{st}	$2L$	f_1
2^{nd}	L	$2f_1$
3^{rd}	$2L/3$	$3f_1$

Longitudinal Waves in a Gas



- Increased pressure on a volume of gas decreases its volume
- Bulk modulus of elasticity is defined

$$K = -V\frac{dp}{dV}$$

Longitudinal Waves in a Gas

- Equations of state for a gas:
  - Ideal gas law: $pV = NkT$
  - Adiabatic gas law: $pV^{\gamma} = constant$
- In an adiabatic process, no heat is absorbed
  - Absorbing heat would remove mechanical energy from a system
  - Propagation of sound waves through a gas is an example of an adiabatic process
- Bulk modulus calculated from equation of state:

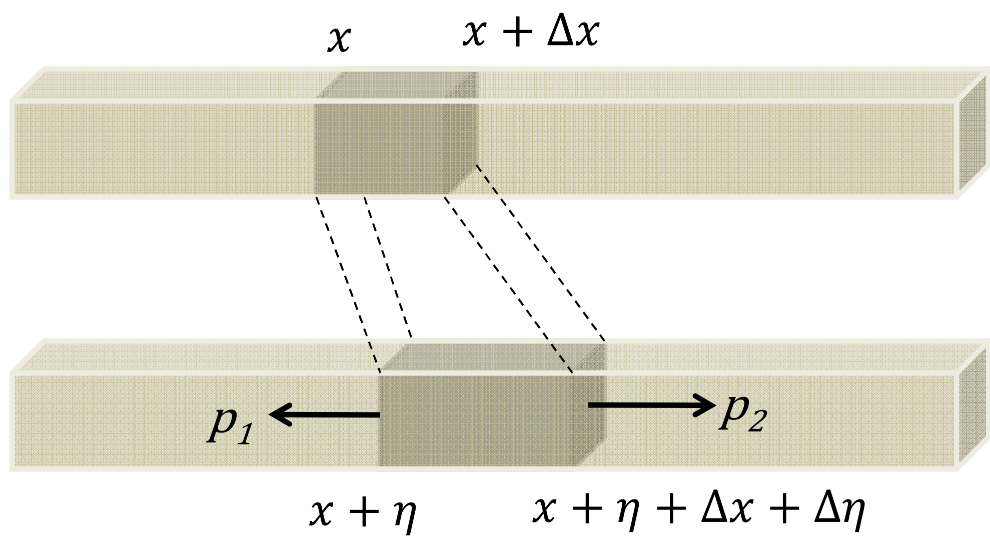
$$V^{\gamma} dp + \gamma p V^{\gamma - 1} dV = 0$$

$$\frac{dp}{dV} = -\gamma p / V$$

$$K = -V \frac{dp}{dV} = \gamma p$$

Longitudinal Waves in a Gas

- By analogy with the solid rod, we consider an element of gas at position x of thickness Δx that is displaced by a distance $\eta(x)$:



Longitudinal Waves in a Gas

- Wave equation:

$$\frac{\partial^2 \eta}{dx^2} = \frac{1}{v^2}\frac{\partial^2 \eta}{\partial t^2}$$

- For a sold rod, $v = \sqrt{Y/\rho}$
- For a gas, $v = \sqrt{K/\rho} = \sqrt{\gamma p/\rho}$
- Changes in pressure and density are very small compared with the average pressure and density.
- At standard temperature and pressure, air has

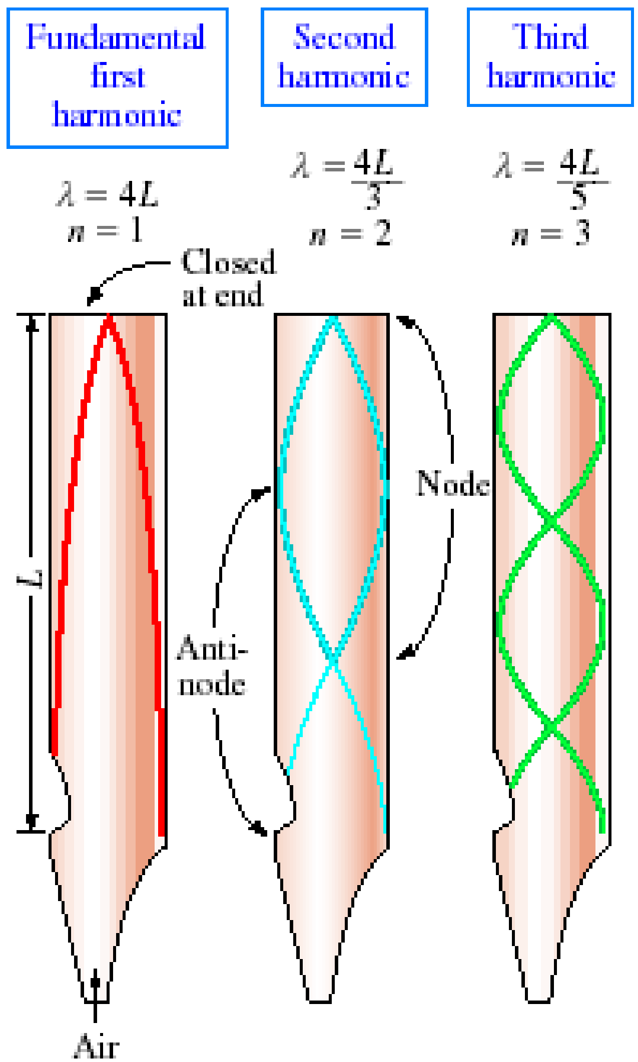
$$\gamma = 1.40$$
$$p = 101.3\ \mathrm{kPa}$$
$$\rho = 1.2\,\mathrm{kg/m^3}$$

$$v = \sqrt{\frac{(1.40)(101.3\times 10^3 N/m^2)}{(1.2\ kg/m^3)}} = 343\ m/s$$

The Physics of Organ Pipes

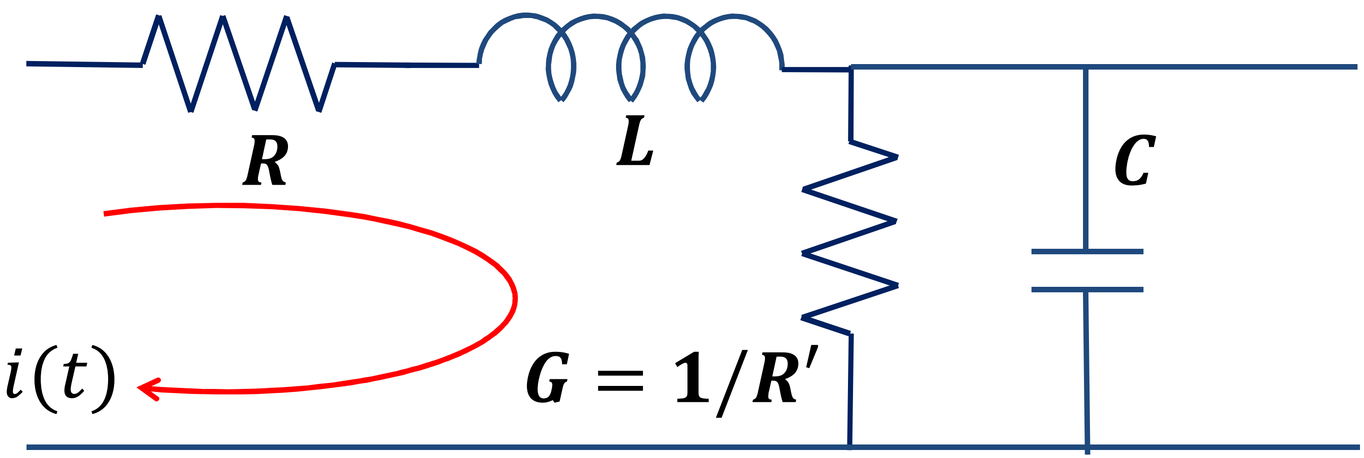
Resonant Cavities

- Air under pressure enters at the bottom
  - Entering air rapidly oscillates between the pipe and the lip
  - The lower end is a pressure anti-node
- Top end can be open or closed
  - Open end is a pressure node/displacement anti-node
  - Closed end a displacement node/pressure anti-node



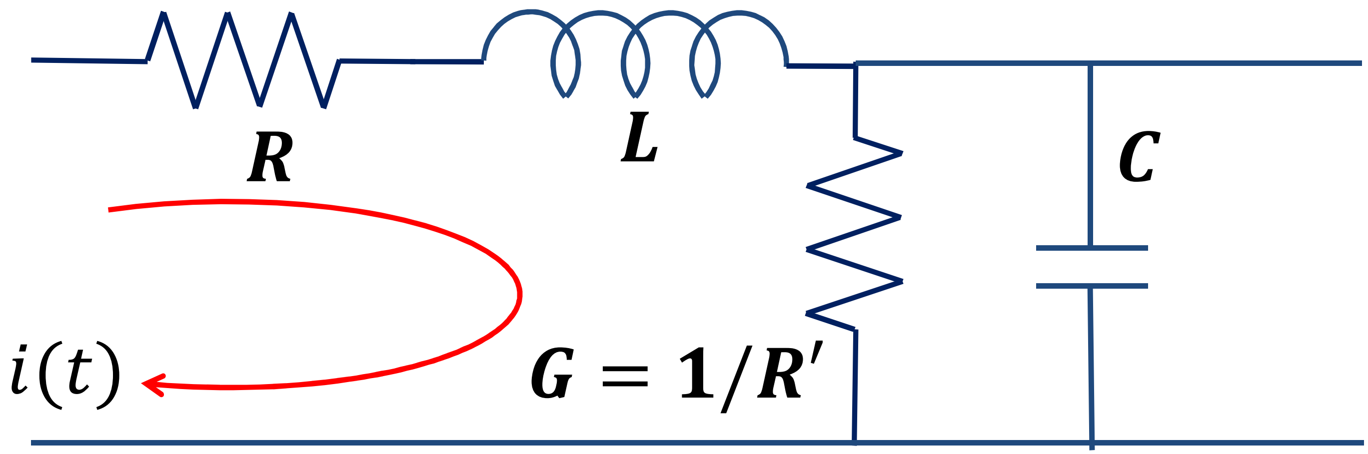
Electrical Circuits

- First, consider one “lump” of a circuit:



- It is convenient to describe the resistor that is in parallel with the capacitor in terms of its conductance, $G = 1/R'$.

Electrical Circuits



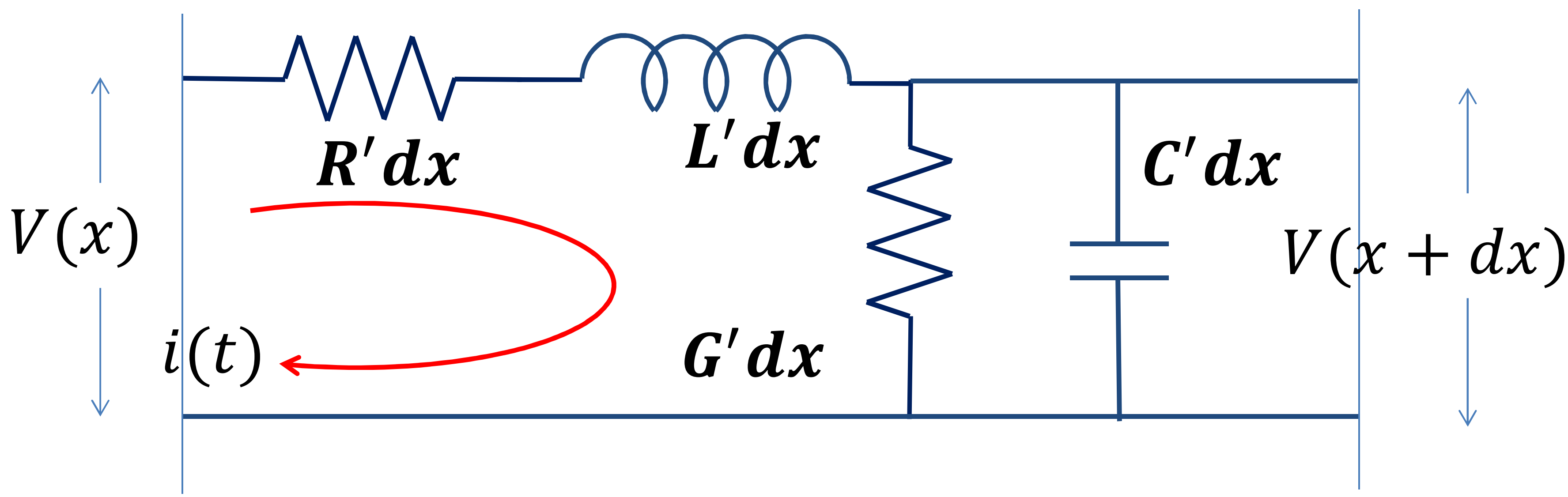
- Calculate the total impedance of the lump:

$$Z_R = R$$
$$Z_L = i\omega L$$
$$Z_C = \frac{1}{i\omega C}$$
$$Z_G = 1/G$$

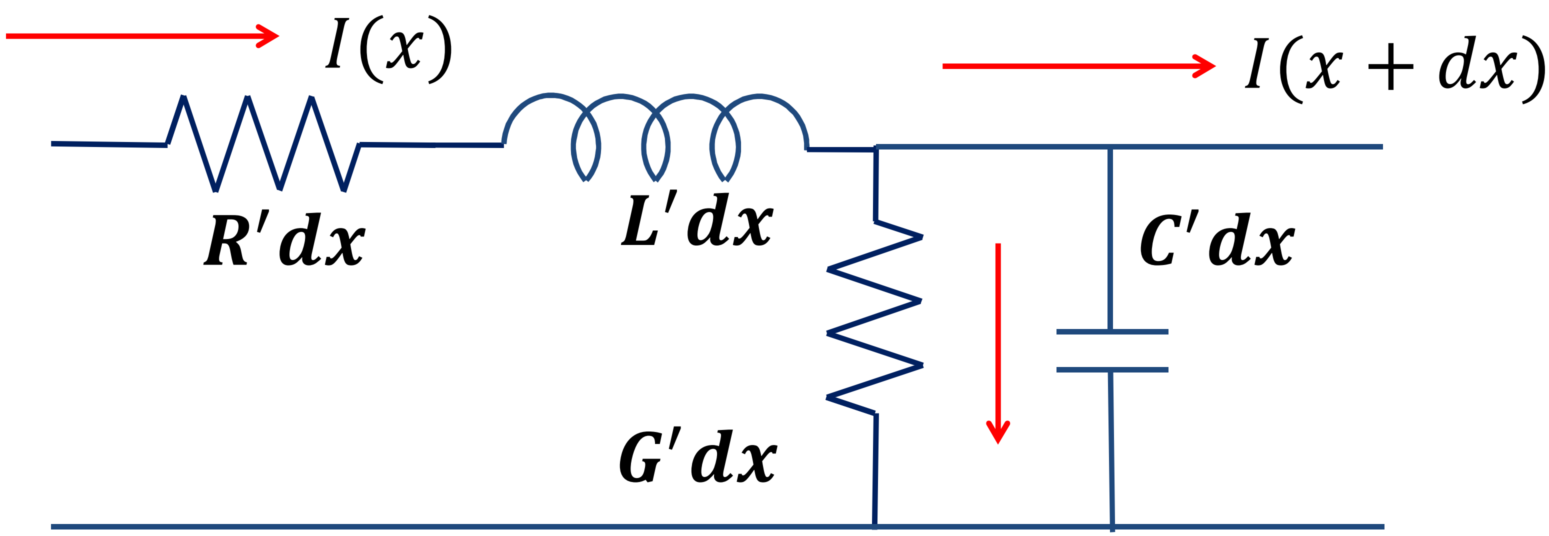
$$X = R + i\omega L$$
$$Y = G + i\omega C$$

Electrical Circuits

- Suppose the resistance, inductance, capacitance and conductance were distributed uniformly with length:
  - Let R' be the resistance per unit length, L' be the inductance per unit length, etc…
- Consider the voltage on either side of the lump:



Electric Circuits



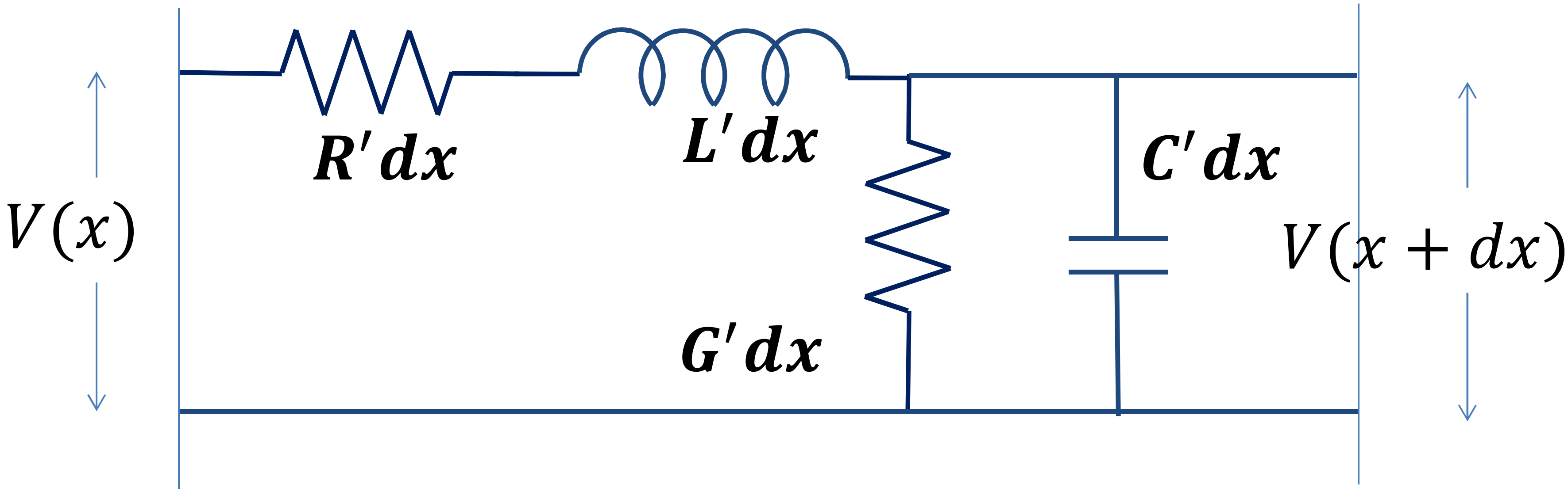
- Current flowing through G' and C' is

$$\Delta I = \frac{V(x)}{Z_{G'+C'}} = V(x)Y$$

$$I(x + dx) = I(x) - V(x)Y$$

$$\frac{\partial I}{\partial x} = \frac{I(x + dx) - I(x)}{dx} = -V(x)Y$$

Electrical Circuits



- Voltage drop across the lump:

$$V(x+dx) = V(x) - I(x)X$$

$$\frac{\partial V}{\partial x} = \frac{V(x+dx) - V(x)}{dx} = -I(x)X$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\partial I}{\partial x}X = XY\,V(x)$$

Electrical Circuits

- When we assume that the voltage is of the form

$$V(x,t) = V(x)e^{i\omega t}$$

$$\frac{\partial^2 V}{\partial t^2} = -\omega^2 V(x)$$

- Using the previous result, $\frac{\partial^2 V}{\partial x^2} = XY\,V(x)$ we get:

$$\frac{\partial^2 V}{\partial x^2} + \frac{XY}{\omega^2}\frac{\partial^2 V}{\partial t^2} = 0$$

- Does this resemble the wave equation?
  - Expand out $XY = (R' + i\omega L')(G' + i\omega C')$
  - When R' and G' are small, which is frequently the case then $XY \approx -\omega^2 L'C'$

Electrical Circuits

- Wave equation:

$$\frac{\partial^2 V}{\partial x^2} = L'C' \frac{\partial^2 V}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 V}{\partial t^2}$$

- Speed of wave propagation is

$$v = \frac{1}{\sqrt{L'C'}}$$