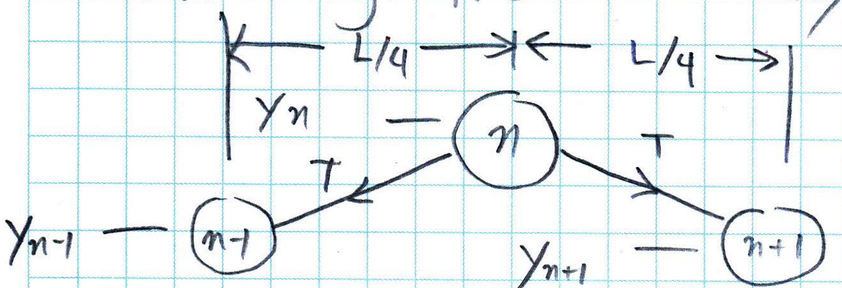
Assignment #5

Let y_n be the vertical displacement of mass n. The forces on mass n are obtained by considering the free body diagram:



Vertical force on mass n is

$$F_n = -T\left(\frac{y_n - y_{n-1}}{L/4}\right) - T\left(\frac{y_n - y_{n+1}}{L/4}\right)$$

$$= -\frac{4T}{L}\left(2y_n - y_{n-1} - y_{n+1}\right)$$

Equations of motion:

$$\frac{M}{3}\ddot{y}_n + \frac{4T}{L}\left(2y_n - y_{n-1} - y_{n+1}\right) = 0$$

$$\text{or} \quad \ddot{y}_n + \omega_0^2\left(2y_n - y_{n-1} - y_{n+1}\right) = 0$$

where $\omega_0^2 = \dfrac{12T}{ML}$

The equations of motion for the three masses are

$$\ddot{y}_1 + 2\omega_0^2 y_1 - \omega_0^2 y_2 = 0$$

$$\ddot{y}_2 + 2\omega_0^2 y_2 - \omega_0^2 y_1 - \omega_0^2 y_3 = 0$$

$$\ddot{y}_3 + 2\omega_0^2 y_3 - \omega_0^2 y_2 = 0$$

Suppose solutions are of the form

$$y_i(t) = a_i \cos \omega t$$

Then $\ddot{y}_n(t) = -\omega^2 y_n(t)$ and the equations of motion are

$$\begin{pmatrix} -\omega^2+2\omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & -\omega^2+2\omega_0^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & -\omega^2+2\omega_0^2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 0$$

Frequencies of normal modes are the eigenvalues.

Let $\lambda = \omega^2$. Then the determinant is

$$(-\lambda+2\omega_0^2)\left[(-\lambda+2\omega_0^2)(-\lambda+2\omega_0^2) - \omega_0^4\right] - \omega_0^4(-\lambda+2\omega_0^2) = 0$$

$$(-\lambda+2\omega_0^2)\left[\lambda^2 - 4\lambda\omega_0^2 + 4\omega_0^4 - \omega_0^4 - \omega_0^4\right] = 0$$

$$(-\lambda+2\omega_0^2)\left[\lambda^2 - 4\lambda\omega_0^2 + 2\omega_0^4\right] = 0$$

Roots are $\lambda = 2\omega_0^2$

$$\lambda = 2\omega_0^2 \pm \sqrt{4\omega_0^4 - 2\omega_0^4}$$

$$= \omega_0^2\left(2 \pm \sqrt{2}\right)$$

$$\omega_1^2 = \omega_0^2(2-\sqrt{2})$$

$$\omega_2^2 = 2\omega_0^2$$

$$\omega_3^2 = \omega_0^2(2+\sqrt{2})$$

These frequencies can be compared with those obtained using the general formula:

$$\omega_n = 2\omega_0 \sin\left(\frac{n\pi}{2(N+1)}\right)$$

where $N = 3$ and $\omega_0^2 = \frac{12T}{ML}$.

This gives $\omega_1 = 2\omega_0 \sin\left(\frac{\pi}{8}\right) = \omega_0\sqrt{2-\sqrt{2}}$

$\omega_2 = 2\omega_0 \sin\left(\frac{\pi}{4}\right) = \omega_0\sqrt{2}$

$\omega_3 = 2\omega_0 \sin\left(\frac{3\pi}{8}\right) = \omega_0\sqrt{2+\sqrt{2}}$

Next, we compare these to the frequencies of the normal modes of a continuous string of length L and total mass M.

These are $\omega'_n = \frac{n\pi}{L}\sqrt{\frac{T}{\mu}} = \frac{n\pi}{L}\sqrt{\frac{LT}{M}} = n\pi\sqrt{\frac{T}{LM}}$

This can be written in terms of $\omega_0 = \sqrt{\frac{12T}{ML}}$ as follows:

$$\omega'_n = \frac{n\pi\omega_0}{\sqrt{12}}.$$

Numerically, the comparison is as follows:

n	ω_n		ω'_n	
1	$\omega_0\sqrt{2-\sqrt{2}}$	$= 0.7654\,\omega_0$	$\pi\omega_0/\sqrt{12}$	$= 0.9069\,\omega_0$
2	$\omega_0\sqrt{2}$	$= 1.414\,\omega_0$	$2\pi\omega_0/\sqrt{12}$	$= 1.814\,\omega_0$
3	$\omega_0\sqrt{2+\sqrt{2}}$	$= 1.848\,\omega_0$	$3\pi\omega_0/\sqrt{12}$	$= 2.721\,\omega_0$

The ratios of these frequencies are as follows:

n	ω_n / ω_1'
1	0.844
2	1.56
3	2.04

which agrees with the solution given in the text.

2. The cavity length is L and the waves move with velocity $v = c$. The wavelengths of the normal modes are

$$\lambda_n = \frac{2L}{n}$$

and the frequencies are

$$\nu_n = \frac{v}{\lambda_n} = \frac{nc}{2L}$$

(b) If $L = 1.5\,m$ we need to count the number of normal modes that have frequencies between $\nu - \Delta\nu$ and $\nu + \Delta\nu$.

(The diagram is misleading here as it suggests that the total width is $\Delta\nu$ rather than $2\Delta\nu$.)

The lowest normal mode is

$$n_{min} = \frac{2L\nu_{min}}{c} = \frac{2(1.5m)(5\times10^{14}s^{-1} - 10^{9}s^{-1})}{3\times10^{8}\ m/s}$$
$$= 4999990$$

The highest normal mode is

$$n_{max} = \frac{2L\nu_{max}}{c} = \frac{2(1.5m)(5\times10^{14}s^{-1} + 10^{9}s^{-1})}{3\times10^{8}\ m/s}$$
$$= 5000010.$$

So there are 21 normal modes excited.
(You would count only 20 if you used $c = 2.998\times10^{8}m/s$ which is also okay.)

To calculate the largest L that will excite only one normal mode, we can consider the density of normal modes,

$$\frac{\Delta n}{\Delta \nu} = \frac{dn}{d\nu} = \frac{2L}{c}$$

If we want $\Delta n = 1$ when $\Delta \nu = 10^9$ Hz, then,

$$L = \frac{c}{2}\frac{\Delta n}{\Delta \nu} = \frac{3\times10^8 \text{ m/s}}{2\times10^9 \text{ s}^{-1}} = 0.15 \text{ m}.$$

The choice of whether one should use $\Delta \nu = 10^9$ Hz or $\Delta \nu = 2\times10^9$ Hz is somewhat ambiguous. Instead, this calculation could be used to define what we mean by $\Delta \nu$ since the distribution does not, in general, have distinct edges.