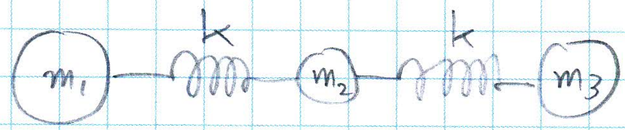
Assignment #4

1.



(a) Force on m_1 is $F_1 = -k(x_1 - x_2)$
Force on m_2 is $F_2 = -2kx_2 + kx_1 + kx_2$
Force on m_3 is $F_3 = -k(x_3 - x_2)$

Equations of motion:

$$m_1 \ddot{x}_1 + k(x_1 - x_2) = 0$$

$$m_2 \ddot{x}_2 + 2kx_2 - kx_1 - kx_3 = 0$$

$$m_3 \ddot{x}_3 + k(x_3 - x_2) = 0$$

If we write $\omega_0^2 = \frac{k}{m_1} = \frac{k}{m_3}$ and $\omega_0'^2 = \frac{k}{m_2}$

then these are

$$\ddot{x}_1 + \omega_0^2 (x_1 - x_2) = 0$$

$$\ddot{x}_2 + 2\omega_0'^2 x_2 - \omega_0'^2 (x_1 + x_3) = 0$$

$$\ddot{x}_3 + \omega_0^2 (x_3 - x_2) = 0$$

Let $\vec{x} = (x_1, x_2, x_3)$. Then, if we write $\vec{x} = \vec{u} \cos \omega t$ then $\ddot{\vec{x}} = -\omega^2 \vec{x}$ and the system of equations is

$$(-\omega^2 + \omega_0^2) x_1 - \omega_0^2 x_2 = 0$$

$$-\omega_0'^2 x_1 + (-\omega^2 + 2\omega_0'^2) x_2 - \omega_0'^2 x_3 = 0$$

$$-\omega_0^2 x_2 + (-\omega^2 + \omega_0^2) x_3 = 0$$

or

$$\begin{pmatrix} -\omega^2 + \omega_0^2 & -\omega_0^2 & 0 \\ -\omega_0'^2 & (-\omega^2 + 2\omega_0'^2) & -\omega_0'^2 \\ 0 & -\omega_0^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

If we write $\lambda = \omega^2$ then we need to solve

$$\begin{vmatrix} \lambda - \omega_0^2 & \omega_0^2 & 0 \\ \omega_0'^2 & \lambda - 2\omega_0'^2 & \omega_0'^2 \\ 0 & \omega_0^2 & \lambda - \omega_0^2 \end{vmatrix} = 0$$

The determinant is

$$(\lambda - \omega_0^2)\left[(\lambda - 2\omega_0'^2)(\lambda - \omega_0^2) - \omega_0^2 \omega_0'^2 \right] - \omega_0^2 \omega_0'^2 (\lambda - \omega_0^2) = 0$$

$$(\lambda - \omega_0^2)\left[(\lambda - 2\omega_0'^2)(\lambda - \omega_0^2) - 2\omega_0^2 \omega_0'^2 \right] = 0$$

$$(\lambda - \omega_0^2)\left[\lambda^2 - \lambda(\omega_0^2 + 2\omega_0'^2) + 2\omega_0'^2\omega_0^2 - 2\omega_0'^2 \omega_0^2 \right] = 0$$

$$(\lambda - \omega_0^2) \cdot \lambda \cdot (\lambda - \omega_0^2 - 2\omega_0'^2) = 0$$

So the roots are

$$\lambda_0 = 0$$
$$\lambda_1 = \omega_0^2$$
and
$$\lambda_2 = 2\omega_0'^2 + \omega_0^2$$

The eigenvectors are determined as follows:
For $\lambda = 0$,

$$\begin{pmatrix} -\omega_0^2 & \omega_0^2 & 0 \\ \omega_0'^2 & -2\omega_0'^2 & \omega_0'^2 \\ 0 & \omega_0^2 & -\omega_0^2 \end{pmatrix} \begin{pmatrix} u_1^{(0)} \\ u_2^{(0)} \\ u_3^{(0)} \end{pmatrix} = 0$$

So $u_1 = u_2$ and $u_2 = u_3$. In this case all three masses are translated in the same direction and they do not oscillate. This corresponds to the uniform motion of the entire system.

When $\lambda = \omega_0^2$

$$\begin{pmatrix} 0 & -\omega_0^2 & 0 \\ -\omega_0'^2 & 2\omega_0'^2 - \omega_0^2 & -\omega_0'^2 \\ 0 & -\omega_0^2 & 0 \end{pmatrix} \begin{pmatrix} u_1^{(1)} \\ u_2^{(1)} \\ u_3^{(1)} \end{pmatrix} = 0$$

Thus, $u_2^{(1)} = 0$ and $u_1^{(1)} = -u_3^{(1)}$.

This normal mode of oscillation is

$$\begin{aligned} x_1(t) &= A\cos(\omega_0 t + \alpha) \\ x_2(t) &= 0 \\ x_3(t) &= -A\cos(\omega_0 t + \alpha) \end{aligned}$$

When $\lambda = 2\omega_0'^2 + \omega_0^2$,

$$\begin{pmatrix} -2\omega_0'^2 & -\omega_0^2 & 0 \\ -\omega_0'^2 & \omega_0^2 & -\omega_0'^2 \\ 0 & -\omega_0^2 & -2\omega_0'^2 \end{pmatrix} \begin{pmatrix} u_1^{(2)} \\ u_2^{(2)} \\ u_3^{(2)} \end{pmatrix} = 0$$

Thus,
$$\begin{aligned} 2\omega_0'^2 u_1 &= -\omega_0^2 u_2 \\ \omega_0^2 u_2 &= -2\omega_0'^2 u_3 \end{aligned}$$

or
$$\begin{aligned} u_1 &= u_3 \\ u_2 &= -\frac{2\omega_0'^2}{\omega_0^2} u_1 \end{aligned}$$

This normal mode of oscillation is

$$\begin{aligned} x_1(t) &= B\cos((2\omega_0'^2 + \omega_0^2)t + \beta) \\ x_2(t) &= -\frac{2\omega_0'^2}{\omega_0^2} B\cos((2\omega_0'^2 + \omega_0^2)t + \beta) \\ x_3(t) &= B\cos((2\omega_0'^2 + \omega_0^2)t + \beta) \end{aligned}$$

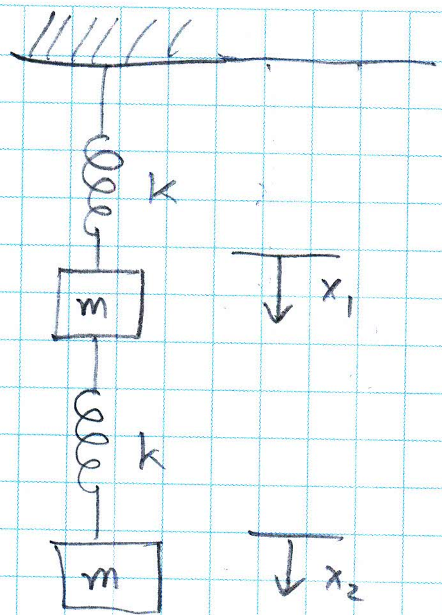
These normal modes of oscillation correspond to symmetric and antisymmetric stretching modes, respectively.

(b) If $m_1 = m_3 = 16$ units and $m_2 = 12$ units, the ratio of the frequencies is

$$\sqrt{\frac{2\omega_0'^2 + \omega_0^2}{\omega_0^2}} = \sqrt{\frac{2\omega_0'^2}{\omega_0^2} + 1} = \sqrt{\frac{2m_1}{m_2} + 1}$$

$$= \sqrt{\frac{2 \times 16}{12} + 1} = \sqrt{\frac{11}{3}}.$$

2.



x_1 and x_2 are measured from their equilibrium positions.

The spring force acting on the lower mass is

$$F_2 = -kx_2 + kx_1$$

The spring force acting on the upper mass is

$$F_1 = -2kx_1 + kx_2$$

Equations of motion:

$$m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 + kx_2 - kx_1 = 0$$

which we can write

$$\ddot{x}_1 + 2\omega_0^2 x_1 - \omega_0^2 x_2 = 0$$

$$\ddot{x}_2 + \omega_0^2 x_2 - \omega_0^2 x_1 = 0$$

where $\omega_0^2 = \frac{k}{m}$.

Let $\vec{x} = \vec{u} \cos \omega t$

then $\ddot{\vec{x}} = -\omega^2 \vec{x}$ and,

$$\begin{pmatrix} -\omega^2 + 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

Let $\lambda = \omega^2$. Then

$$\begin{vmatrix} \lambda - 2\omega_0^2 & \omega_0^2 \\ \omega_0^2 & \lambda - \omega_0^2 \end{vmatrix} = 0$$

$$(\lambda - 2\omega_0^2)(\lambda - \omega_0^2) - \omega_0^4 = 0$$

$$\lambda^2 - 3\omega_0^2 + 2\omega_0^4 - \omega_0^4 = 0$$

$$\lambda^2 - 3\lambda\omega_0^2 + \omega_0^4 = 0$$

$$\lambda = \frac{3\omega_0^2 \pm \sqrt{9\omega_0^4 - 4\omega_0^4}}{2}$$

$$\lambda = \omega^2 = \omega_0^2 \left(\frac{3}{2} \pm \frac{\sqrt{5}}{2} \right) = (3 \pm \sqrt{5}) \cdot \frac{k}{2m}.$$

The ratio of frequencies is $\sqrt{\dfrac{3+\sqrt{5}}{3-\sqrt{5}}} = \sqrt{\dfrac{(3+\sqrt{5})^2}{3^2-5}}$

$$= \frac{3+\sqrt{5}}{2}.$$

This turns out to be equal to $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$

as can be verified:

$$\frac{\sqrt{5}+1}{\sqrt{5}-1} = \frac{(\sqrt{5}+1)(\sqrt{5}+1)}{(\sqrt{5}+1)(\sqrt{5}-1)} = \frac{(\sqrt{5}+1)^2}{5-1} = \frac{5+2\sqrt{5}+1}{4}$$

$$= \frac{3+\sqrt{5}}{2}.$$

To find the ratio of amplitudes, we need to determine the eigenvectors.

When $\omega^2 = \frac{\omega_0^2}{2}(3+\sqrt{5})$,

$$\frac{\omega_0^2}{2}\begin{pmatrix} -3-\sqrt{5}+4 & -2 \\ -2 & -3-\sqrt{5}+2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

So $(1-\sqrt{5})u_1 = 2u_2$

and the ratio of amplitudes is $\frac{u_2}{u_1} = \frac{1-\sqrt{5}}{2}$

When $\omega^2 = \frac{\omega_0^2}{2}(3-\sqrt{5})$,

$$\frac{\omega_0^2}{2}\begin{pmatrix} -3+\sqrt{5}+4 & -2 \\ -2 & -3+\sqrt{5}+2 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

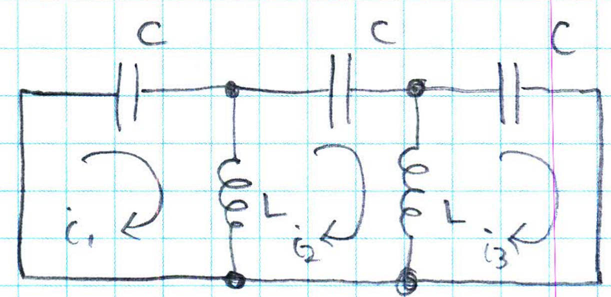
So $(1+\sqrt{5})u_1 = 2u_2$

and the ratio of amplitudes is

$$\frac{u_2}{u_1} = \frac{1+\sqrt{5}}{2}.$$

(8)

3.



Loop with i_1:

$$-\frac{1}{C}\int i_1(t)\,dt - L\left(\frac{di_1}{dt} - \frac{di_2}{dt}\right) = 0$$

Loop with i_2:

$$-\frac{1}{C}\int i_2(t)\,dt - L\left(\frac{di_2}{dt} - \frac{di_3}{dt}\right) - L\left(\frac{di_2}{dt} - \frac{di_1}{dt}\right) = 0$$

Loop with i_3:

$$-\frac{1}{C}\int i_3(t) - L\left(\frac{di_3}{dt} - \frac{di_2}{dt}\right) = 0$$

Differentiate these once with respect to time and write $\omega_0^2 = \frac{1}{LC}$:

(a) $$\frac{d^2 i_1}{dt^2} + \omega_0^2 i_1 - \frac{d^2 i_2}{dt^2} = 0$$

$$2\frac{d^2 i_2}{dt^2} - \frac{d^2 i_1}{dt^2} - \frac{d^2 i_3}{dt^2} + \omega_0^2 i_2 = 0$$

$$\frac{d^2 i_3}{dt^2} + \omega_0^2 i_3 - \frac{d^2 i_2}{dt^2} = 0$$

(b) If we write $i(t) = I e^{i\omega t}$ then $\frac{d^2 i}{dt^2} = -\omega^2 i$ and the system of equations is

$$\begin{pmatrix} -\omega^2 + \omega_0^2 & \omega^2 & 0 \\ \omega^2 & -2\omega^2 + \omega_0^2 & \omega^2 \\ 0 & \omega^2 & -\omega^2 + \omega_0^2 \end{pmatrix} \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} = 0$$

The frequencies of the normal modes are λ :

$$\begin{vmatrix} -\lambda + \omega_0^2 & \lambda & 0 \\ \lambda & -2\lambda + \omega_0^2 & \lambda \\ 0 & \lambda & -\lambda + \omega_0^2 \end{vmatrix} = 0$$

$$\begin{vmatrix} \lambda - \omega_0^2 & -\lambda & 0 \\ -\lambda & 2\lambda - \omega_0^2 & -\lambda \\ 0 & -\lambda & \lambda - \omega_0^2 \end{vmatrix} = 0$$

$$(\lambda - \omega_0^2)\left[(2\lambda - \omega_0^2)(\lambda - \omega_0^2) - \lambda^2\right] - \lambda^2(\lambda - \omega_0^2) = 0$$

$$(\lambda - \omega_0^2)\left[(2\lambda - \omega_0^2)(\lambda - \omega_0^2) - 2\lambda^2\right] = 0$$

$$(\lambda - \omega_0^2)\left[-3\lambda\omega_0^2 + \omega_0^4\right] = 0$$

Therefore, $\lambda = \omega_0^2$ or $\lambda = \frac{\omega_0^2}{3}$.

In fact, there are only two normal modes because the three currents are not independent. To see this, add the first and third equation. The result is a constant multiple of the second equation.