Purdue University Department of Physics

Physics 24100

Electricity & Optics

Lecture 4 – Chapter 22 sec. 2-3

Fall 2012 Semester

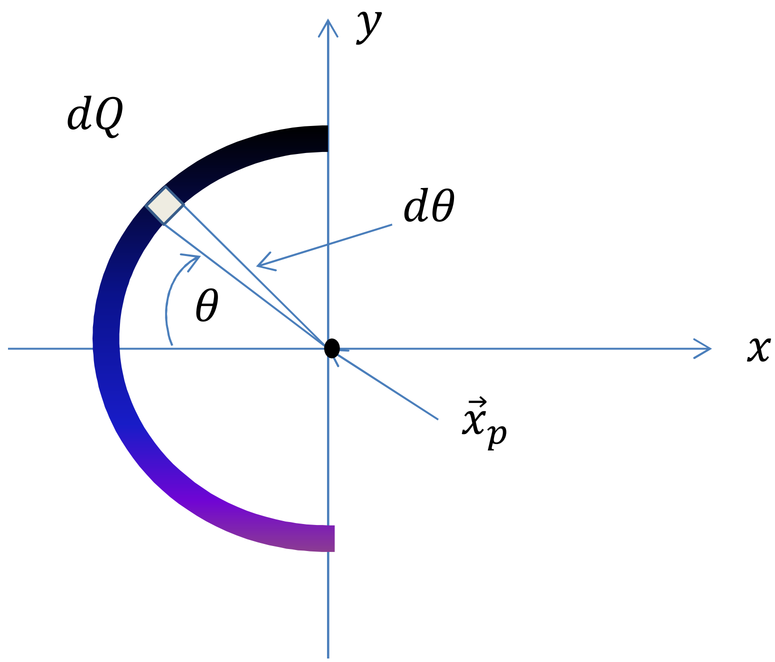
Matthew Jones

Tuesday's Clicker Question

- Charge is uniformly distributed on a semicircular ring with radius a.
- The linear charge density is λ.
- We know that

$$\vec{E}(\vec{x}_p) = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{r^2}\hat{r}$$

- What is $\dfrac{dQ}{r^2}$ when $\vec{x}_p$ is at the origin?



(a) $\dfrac{\lambda}{a}d\theta$

(b) $\lambda\, a\, d\theta$

(c) $\dfrac{\lambda\, d\theta}{a^2}$

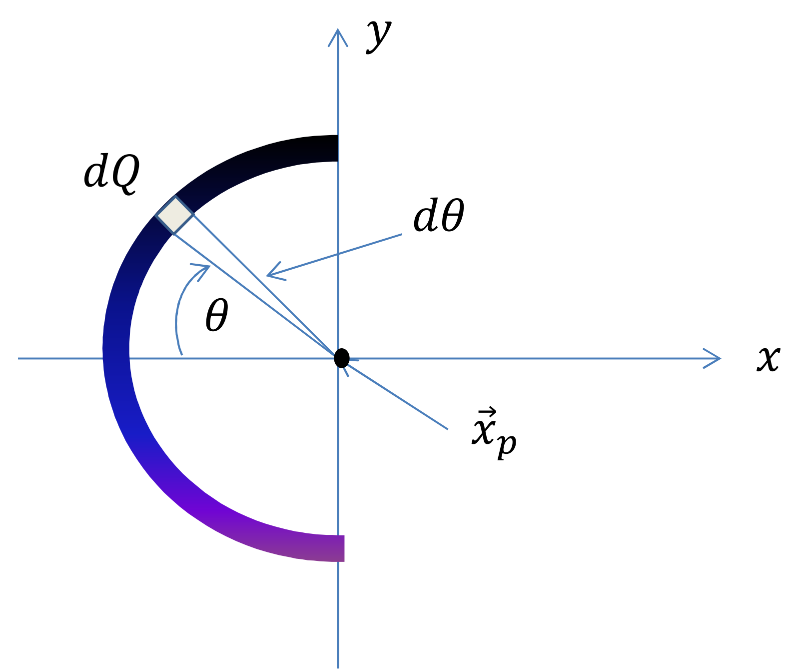
(d) $\pi\, a\, \lambda$

Tuesday's Clicker Question

$$\vec{E}(\vec{x}_p) = \frac{1}{4\pi\epsilon_0}\int \frac{dQ}{r^2}\hat{r}$$

- The linear charge density is λ.
- Charge element: $dQ = \lambda\, ds$
- Element of arc length: $ds = a\, d\theta$
- Therefore, $dQ = \lambda\, a\, d\theta$
- The distance is constant, $r = a$
- Thus,

$$\frac{dQ}{r^2} = \frac{\lambda\, a\, d\theta}{a^2} = \frac{\lambda\, d\theta}{a}$$



(a) $\frac{\lambda}{a}d\theta$

(b) $\lambda\, a\, d\theta$

(c) $\frac{\lambda\, d\theta}{a^2}$

(d) $\pi\, a\, \lambda$

Summary of Physics Concepts

- Fundamental concepts:
  - Coulomb's law: $\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{Q_1 Q_2}{r^2} \hat{r}$
  - Electric field: $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r} \Rightarrow \vec{F} = q\,\vec{E}$
  - Principle of superposition: $\vec{E} = \vec{E}_1 + \vec{E}_2 + \ \ldots$
- Connection with mechanics:
  - Newton's second law: $\vec{F} = m\,\vec{a}$
- Definitions and derived quantities:
  - Electric dipole, potential energy in an $\vec{E}$ field.
  - Charge density: linear, surface, volume

Technical Concepts

- We also used some mathematical techniques:
  - We used a lot of vectors…
  - Series expansion:

$$\frac{1}{z+a} = \frac{1}{z}(1+a/z)^{-1} = \frac{1}{z}\left(1 - \frac{a}{z} + \frac{a^2}{z^2} - \cdots\right)$$

$$\frac{1}{(z+a)^2} = \frac{1}{z^2}(1+a/z)^{-2} = \frac{1}{z^2}\left(1 - \frac{2a}{z} + \frac{3a^2}{z^2} - \cdots\right)$$

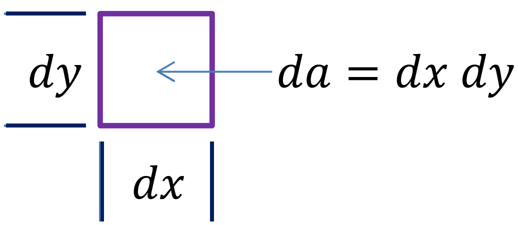
- We solved a differential equation:

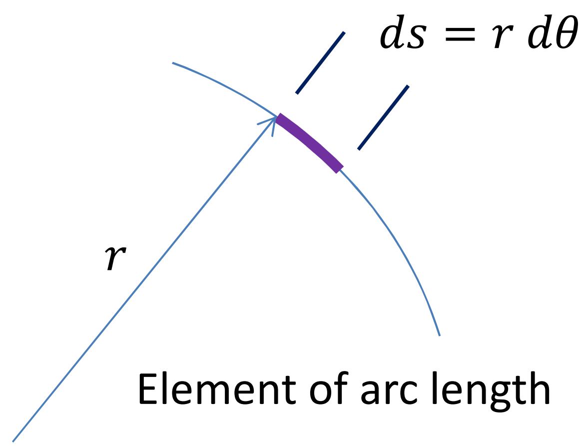
$$\vec{F} = m\vec{a} = m\frac{d^2\vec{x}}{dt^2} \Rightarrow \frac{d^2\vec{x}}{dt^2} = \frac{q\vec{E}}{m}$$

$$\vec{x}(t) = \vec{x}_0 + \vec{v}_0 t + \frac{q\vec{E}}{2m}t^2$$

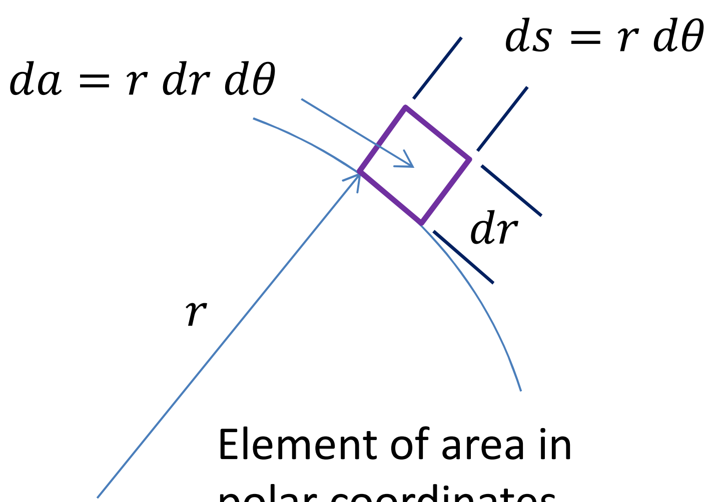
Technical Concepts

- We used some geometrical constructs:
  - Differential elements:





Element of arc length



Element of area in polar coordinates

Technical Concepts

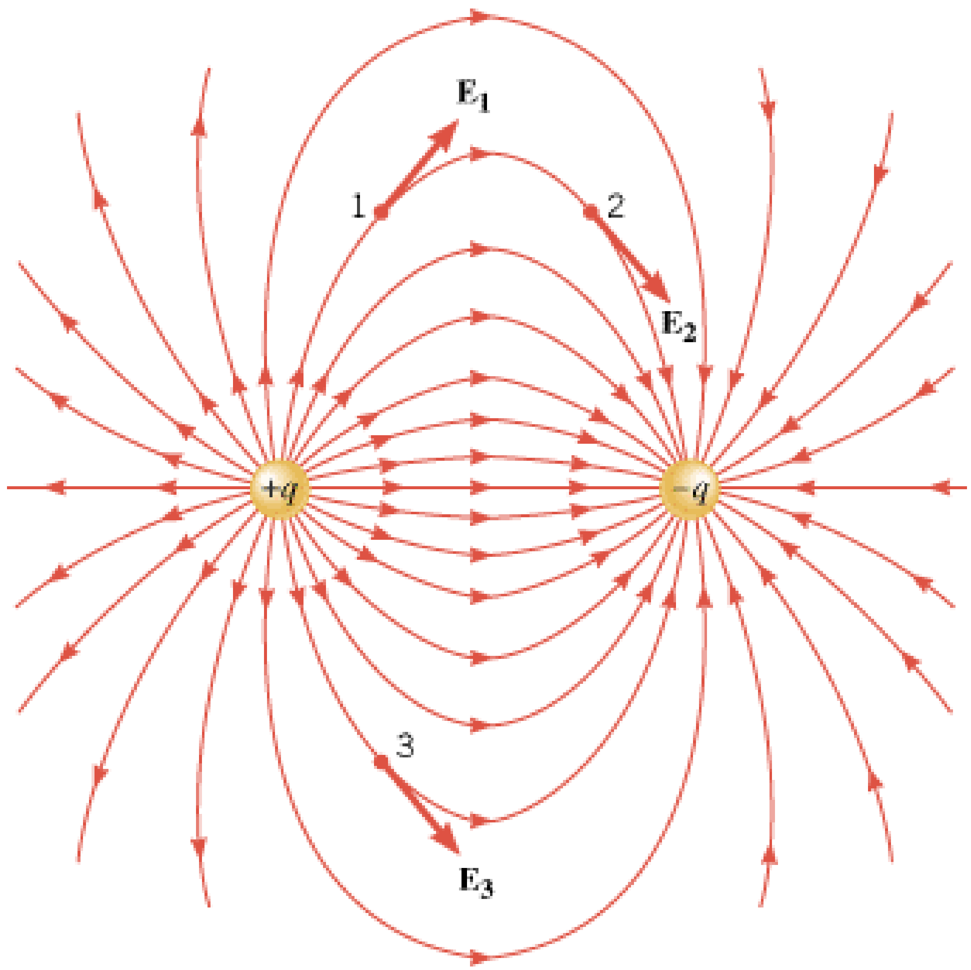
- Then we evaluated some integrals (ugh…)

$$E_x = \frac{\lambda}{4\pi\epsilon_0}\int_{-L/2}^{L/2} \frac{(x-s)ds}{\left((x-s)^2+y^2\right)^{3/2}}$$

$$= \frac{\lambda}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x-L/2)^2+y^2}} - \frac{1}{\sqrt{(x+L/2)^2+y^2}}\right)$$

- These technical skills are useful, but they are distinct from the physical concepts.
  - Try not to confuse the two…
  - You should understand the fundamental physics concepts
  - You should be proficient in basic technical concepts
  - For more complicated problems, refer to notes, reference books, math texts, appendices, google, etc…

Lecture 2 review: Electric Field Lines

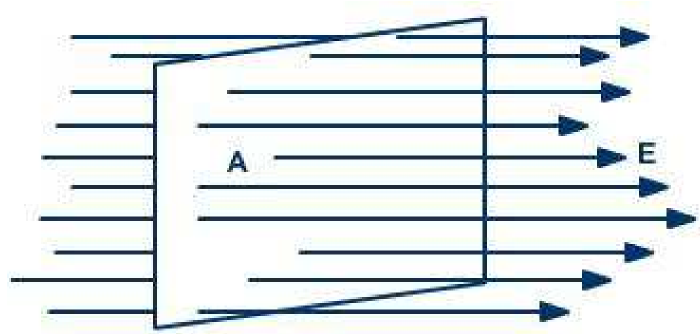


- A positive charge has more lines ***coming out*** than going in.
- A negative charge has more lines ***going in*** than coming out.
- The ***direction*** of the electric field is ***tangent*** to the electric field lines at any point.
- The ***magnitude*** of the electric field is proportional to the ***density*** of electric field lines.

A test charge placed in the electric field will feel a force, $\vec{F} = q_0\vec{E}$.

Electric Flux

- The ***magnitude*** of the electric field is proportional to the ***density*** of electric field lines.
- Consider how many electric field lines cross a small surface with area A:



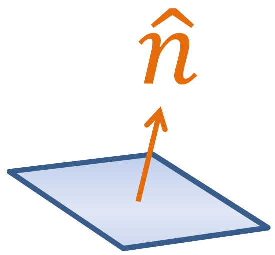
- ***If*** the density is *uniform* over the surface and the surface is *perpendicular* to the $\vec{E}$ field, then we define the flux to be…

$$\phi = EA$$

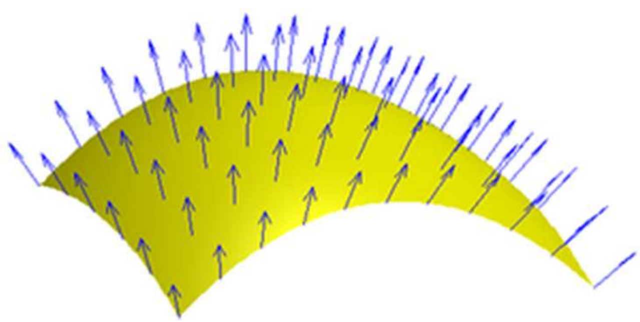
phi is for flux

Electric Flux

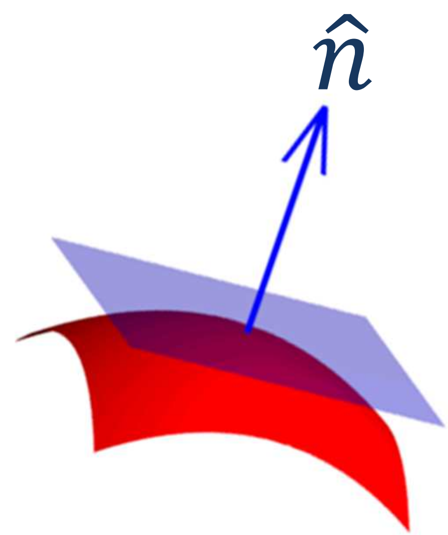
- But, the surface might not be flat…
- The unit vector that is perpendicular to a surface at a given point is called the normal vector at that point.



If the surface is a plane, then the normal vector is the same at all points.



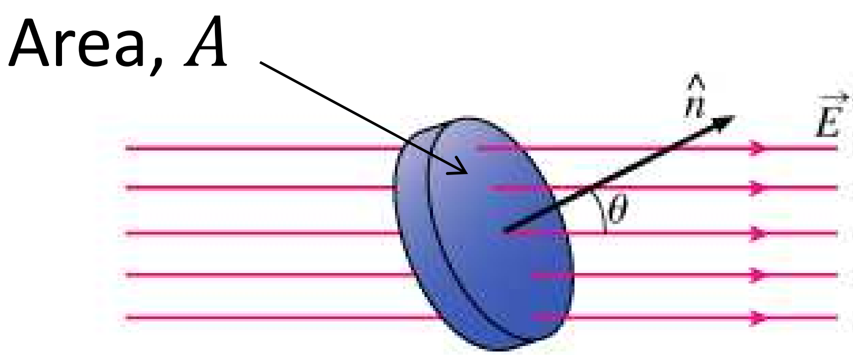
For a curved surface, the direction depends where the point is located.



Electric Flux

- ***If*** the $\vec{E}$ field is *uniform* over a *flat* surface, then the flux is:

$$\phi = A\,\hat{n} \cdot \vec{E} = E\,A\,\cos\theta$$



- But, the $\vec{E}$ field might not be uniform over the whole surface…

Electric Flux

- If the surface is tiny enough, then the $\vec{E}$ field should be approximately uniform.
- The small element of flux through the tiny surface element is:

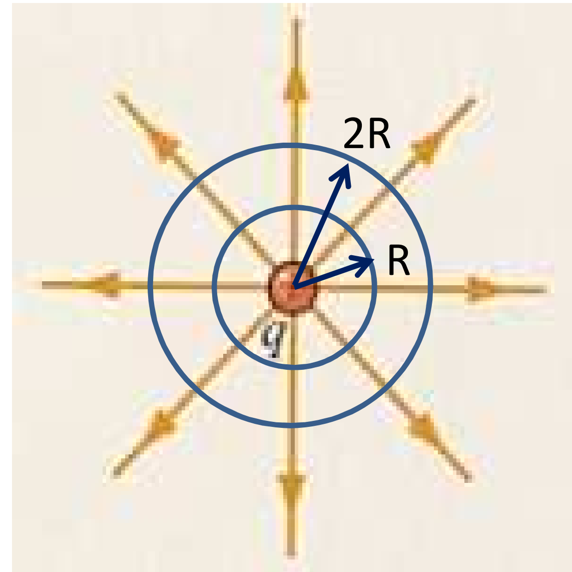
$$\Delta\phi = \hat{n} \cdot \vec{E}\, \Delta A$$

- Then we add up all the tiny flux elements over the whole surface, S, then:

$$\phi_{net} = \int_S \hat{n} \cdot \vec{E}\, dA$$

(general definition of electric flux through a surface, S.)

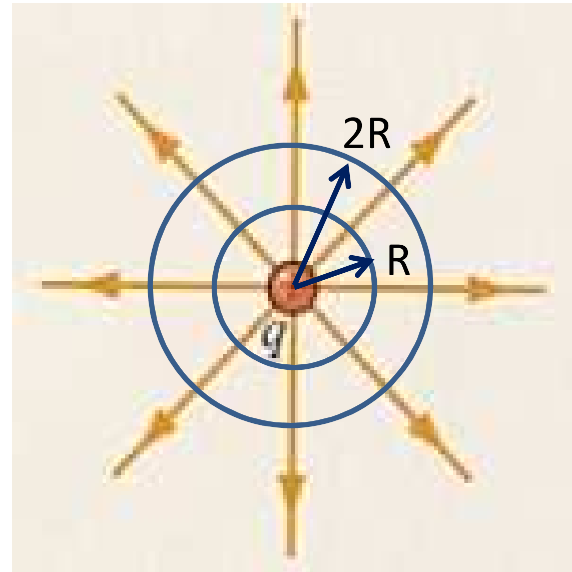
Question



- Consider a point charge, q, at the origin surrounded by two spheres.
  - One has radius R
  - The other has radius $2R$
- What can we say about the net flux through the two surfaces?

(a) $\phi_R < \phi_{2R}$ (b) $\phi_R = \phi_{2R}$ (c) $\phi_R > \phi_{2R}$

Question



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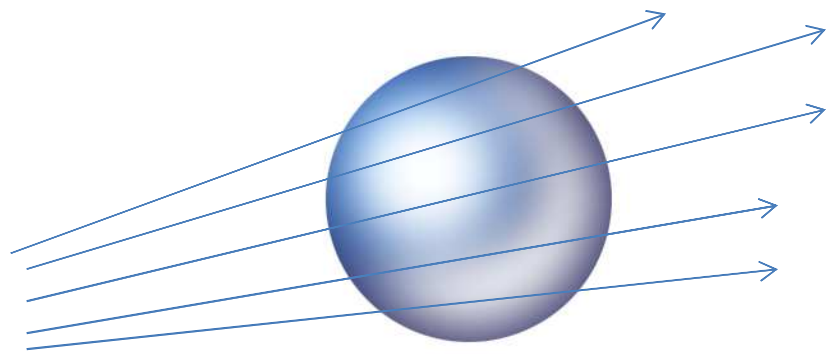
Why Electric Flux?

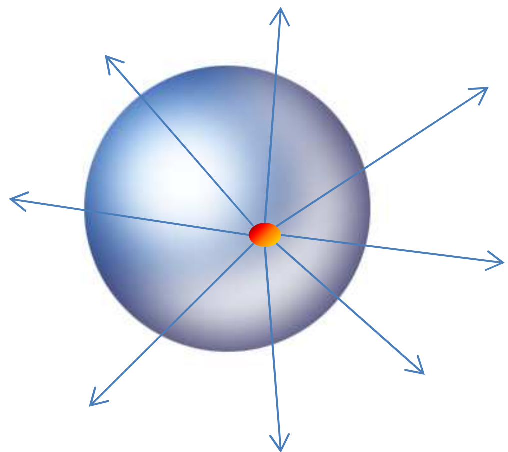
- Consider the electric flux through a closed surface:

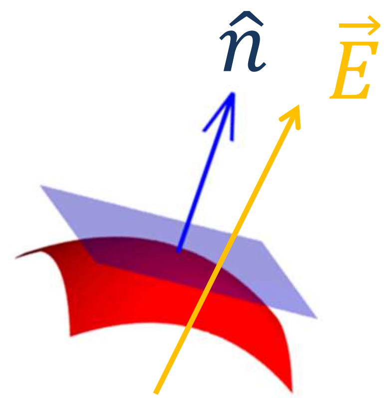


- What is the net flux through the surface?
- If an $\vec{E}$ field line goes in, it must also come out…
- The total flux is exactly zero!
  - Unless there are charges inside…

Why Electric Flux

- What if there is a positive charge inside the surface?





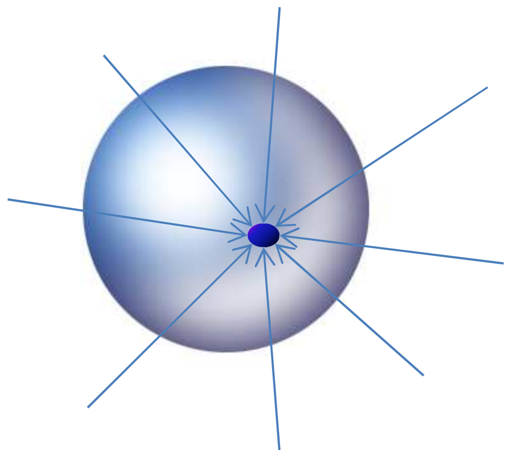
$\cos\theta > 0$

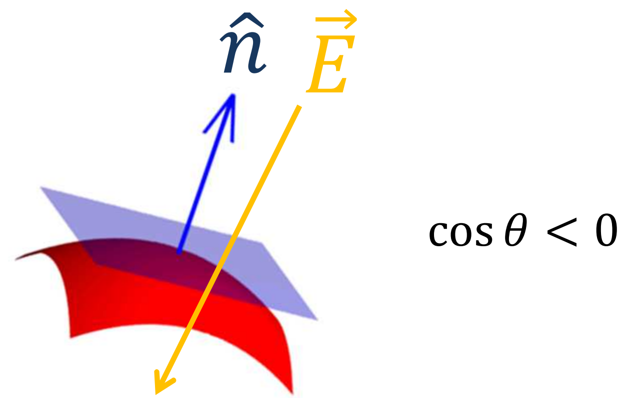
- Then $\hat{n} \cdot \vec{E} > 0$ everywhere on the surface…
- Total flux, $\phi = \oint_S \hat{n} \cdot \vec{E}\, dA$ is positive.

This just means an integral over a closed path or surface, S.

Why Electric Flux

- What if there is a negative charge inside the surface?





- Then $\hat{n} \cdot \vec{E} < 0$ everywhere on the surface…
- Total flux, $\phi = \oint_S \hat{n} \cdot \vec{E}\, dA$ is negative.
- ***The total flux through a closed surface tells us something about the total charge inside.***

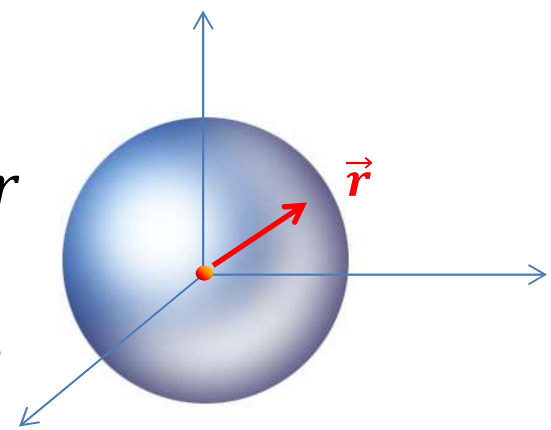
Gauss's Law

$$\phi_{net} = \oint_S \hat{n} \cdot \vec{E}\, dA = \frac{Q_{inside}}{\epsilon_0}$$

- We can often use this to calculate $\vec{E}$…
- We can use *any choice* of "Gaussian surface" if it makes the problem simple.
- It can be much easier than the integrals we did on Tuesday.
- Does this really work? Let's check…

Trivial Example: point charge

- Consider a point charge, Q, at the origin.
- What is $\vec{E}$ at a point, $\vec{r}$?
  - Consider a spherical surface of radius r with the charge at the center.
  - From symmetry, the $\vec{E}$ field will be the same magnitude everywhere on the surface.
  - The $\vec{E}$ field is always perpendicular to the surface.
  - Therefore, $\hat{r} \cdot \vec{E}$ is a constant everywhere on the surface… we can take it out of the integral.

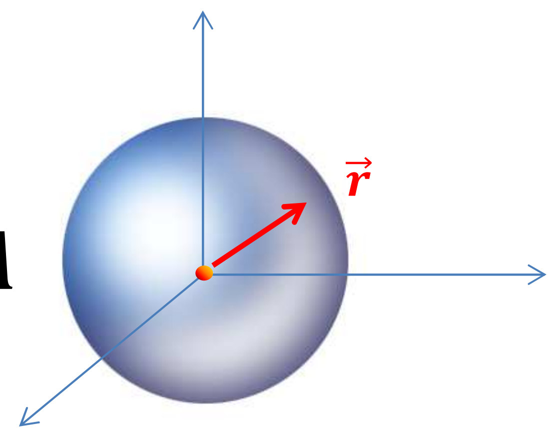


The normal vector, $\hat{n}$ is the same as $\hat{r}$…

Trivial Example: point charge

- Gauss's Law:

$$\phi_{net} = \oint_S \hat{n} \cdot \vec{E}\, dA = \frac{Q_{inside}}{\epsilon_0}$$



- But, $\oint_S \hat{n} \cdot \vec{E}\, dA = \hat{n} \cdot \vec{E} \oint_S dA$
- But, $\oint_S dA$ is just the surface area of a sphere: $A = 4\pi r^2$.
- So, $4\pi r^2 \hat{r} \cdot \vec{E} = \frac{Q}{\epsilon_0}$

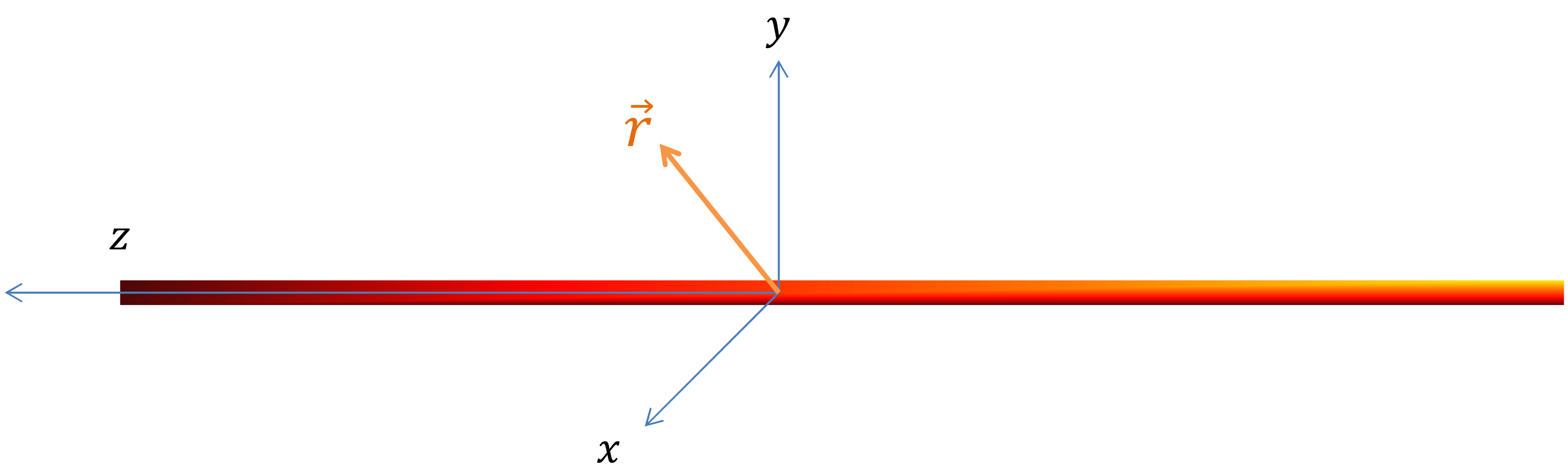
$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2}\, \hat{r}$$

Gauss’s Law

- Gauss’s law works for the trivial example of a point charge, but it is also true in general.
- It is true even for continuous charge distributions.
- Next example: an infinite line of charge.

Question:

- Infinite line with positive charge density, λ, on the z-axis:

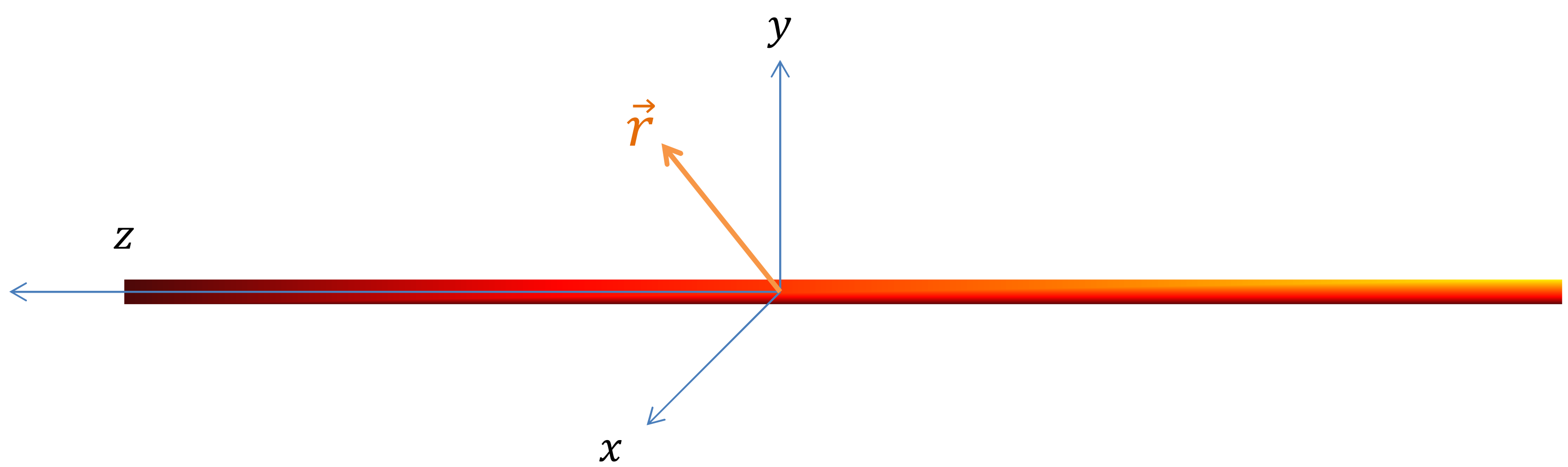


- What direction is $\vec{E}$ pointing at any point $\vec{r}$ in the x-y plane?

(a) $\hat{\imath}$ (b) $\hat{\jmath}$ (c) $\hat{k}$ (d) $\hat{r}$

Question:

- Infinite line with positive charge density, λ, on the z-axis:

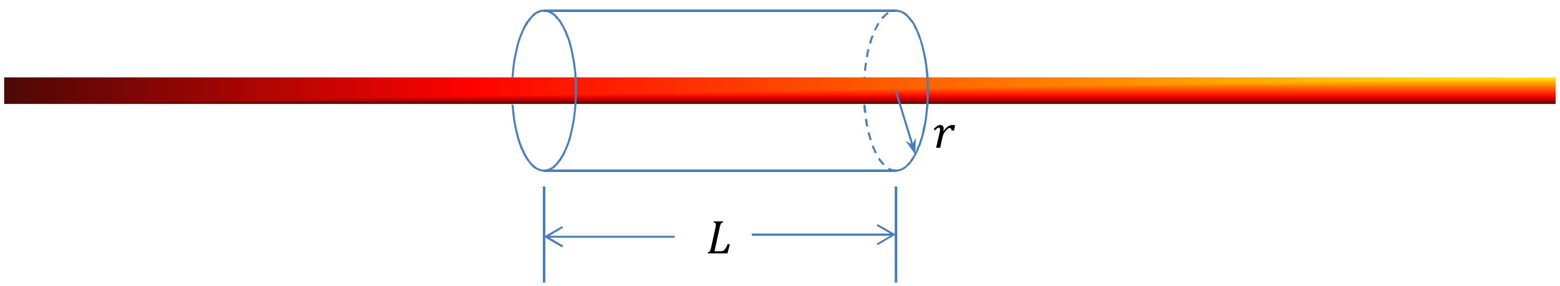


- What direction is $\vec{E}$ pointing at any point $\vec{r}$ in the x-y plane?

(a) $\hat{\imath}$ (b) $\hat{\jmath}$ (c) $\hat{k}$ (d) $\hat{r}$

Electric field from an infinite line charge

- Consider a Gaussian cylinder of radius r and length L along the line:

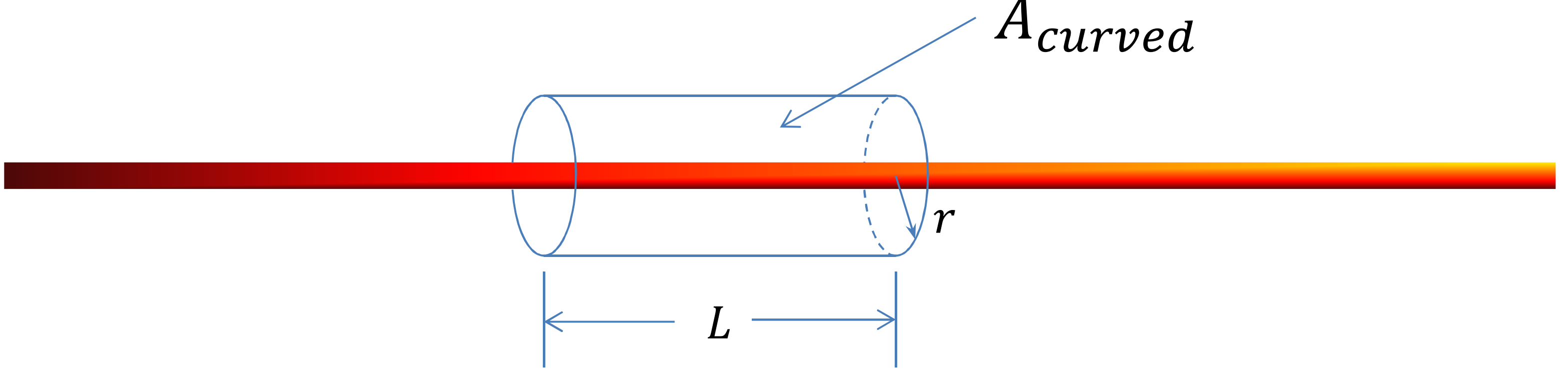


- What is Q_{inside}?

$$Q_{inside} = \lambda L$$

Electric field from an infinite line charge

- Consider a Gaussian cylinder of radius r and length L along the line:

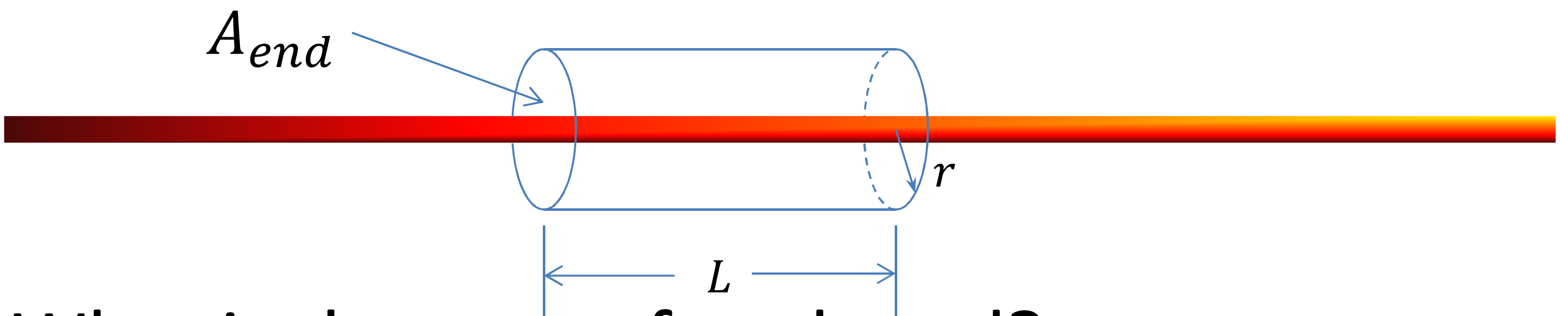


- What is the area of the curved surface?

$$A_{curved} = 2\pi r L$$

Electric field from an infinite line charge

- Consider a Gaussian cylinder of radius r and length L along the line:



- What is the area of each end?

$$A_{end} = \pi r^2$$

- But $\vec{E}$ is perpendicular to the normal vector on the ends of the cylinder: $\hat{k} \cdot \vec{E} = 0$.

Electric field from an infinite line charge

- Gauss's Law: $\phi_{net} = \oint_S \hat{n} \cdot \vec{E}\, dA = \frac{Q_{inside}}{\epsilon_0}$

$$Q_{inside} = \lambda\, L$$

Equals zero…

$$\oint_S \hat{n} \cdot \vec{E}\, dA = \int_{curved} \hat{r} \cdot \overrightarrow{E}\, dA + \int_{ends} \hat{k} \cdot \overrightarrow{E}\, dA$$

$$\int_{curved} \hat{r} \cdot \overrightarrow{E}\, dA = 2\pi r L\, E$$

$$2\pi r L\, E = \frac{\lambda\, L}{\epsilon_0}$$ so $$\vec{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{r}\hat{r}$$

Uniform Spherical Charge Distribution

Electric field is always pointing away from the center of the sphere.

At a fixed radius, $\hat{r} \cdot \vec{E}$ is constant.

If $r < R$ then charge inside radius r is
$Q_{inside} = \frac{4}{3}\pi r^3 \rho$

If $r > R$ then charge inside radius r is
$Q_{inside} = \frac{4}{3}\pi R^3 \rho$

Surface area of sphere of radius r is
$A = 4\pi r^2$

Volume charge density is ρ

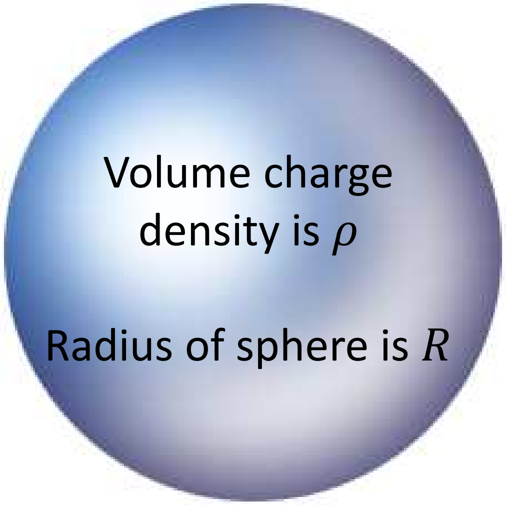
Radius of sphere is R

We can also write
$$\rho = \frac{Q}{V} = \frac{Q}{\frac{4}{3}\pi R^3}$$

Uniform Spherical Charge Distribution

Gauss's Law:

$$\phi_{net} = \oint_S \hat{n} \cdot \vec{E}\, dA = \frac{Q_{inside}}{\epsilon_0}$$

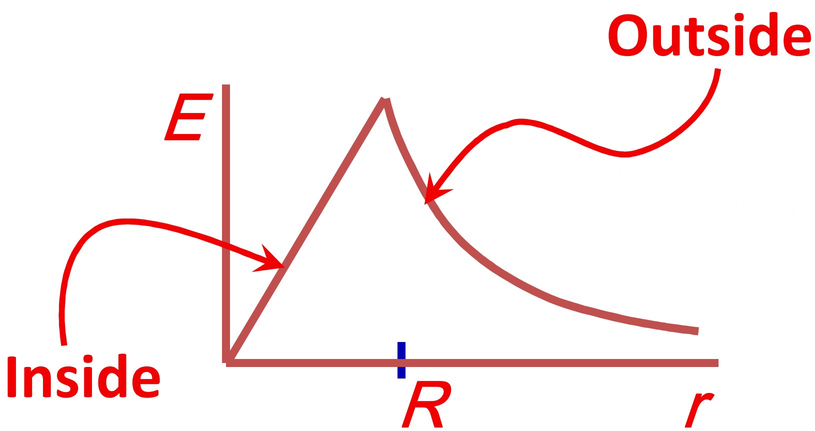


If $r < R$ then $4\pi r^2 E = \frac{4\pi r^3 \rho}{3\epsilon_0}$

$\vec{E} = \frac{r\,\rho}{3\epsilon_0}\hat{r}$ or $\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}\hat{r}$

If $r > R$ then $4\pi r^2 E = \frac{Q}{\epsilon_0}$

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}$$



Question:

- Consider a spherical *shell* of charge with radius R and surface charge density, σ.
- The total charge is $Q = \ \sigma A = 4\pi R^2 \sigma$.
- What is $\vec{E}$ at a point $\vec{r}$ located outside the sphere? $(r > R)$

(a) $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2} \hat{r}$ (b) $\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$ (c) $\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{Q}{r} \hat{r}$