Faraday Effect in Water

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In this paper I give an outline of my research into the Faraday effect in water. I will show that the dependence of the angle of rotation of the plane of polarization is linearly dependent upon the strength of the magnetic field. Furthermore, I will derive the Verdet constant of water.

I. INTRODUCTION

In 1845, Michael Faraday forever changed the way we think about light. In his original experiment, Faraday placed a piece of lead glass in a strong magnetic field and passed a beam of plane polarized light through it. He noted that in the presence of a magnetic field the angle of the plane of polarization changed. The rotation of the plane of polarization of plane polarized light as it is transmitted through an isotropic medium parallel to a magnetic field is now called the Faraday effect. This monumental experiment bridged the chasm between electromagnetism and optics.

Since Faraday's initial experiment, the Faraday effect has been observed in many substances in many different phases. From experiment it has been determined that the Faraday effect is proportional to the magnetic field strength, B, and to the distance the light travels in the medium, l. The angle of rotation of the plane of polarization may be described by the following equation:

$$\theta_R = VBl$$

where V is called the Verdet constant and varies for each substance. The Verdet constant is usually given in $\frac{(\mathrm{min})}{(G)(cm)}$. From the dimensions of the Verdet constant we see that the magnetic field strength is in Gauss and the path length is in centimeters.

II. THEORY

In order to understand the Faraday effect we will assume that the transmission medium is isotropic. Furthermore, we shall assume that the electrons in the medium are harmonically bound to each molecule. The classical equation of the motion of an electron is given by:

$$\ddot{r} + 2b\dot{r} + \frac{k}{m}r = \frac{e}{m}\vec{E}$$

which is measured in m/s^2. b and k are the damping and spring constants, respectively. Solving for $\vec{r}(t)$ we get:

$$\vec{r}(t) = \frac{(e/m)\vec{E}}{(w_o^2 - w^2) - 2ibw}$$

where $w_o = \sqrt{k/m}$ is the natural frequency of the molecule in radians per second. Let there be N molecules per unit volume with frequency w_o. Then the induced dipole moment per unit volume, $\vec{P}$, is:

$$\vec{P} = Ne\vec{r} = \frac{(Ne^2/m)\vec{E}}{(w_o^2 - w^2) - 2ibw}$$

measured in $C \cdot m^{-2}$. The susceptibility, $\chi_E \equiv \frac{P}{\epsilon_o E}$, is then:

$$\chi_E = \frac{Ne^2/m\epsilon_o}{(w_o^2 - w^2) - 2ibw}$$

where ϵ_o is the permittivity of free space. The relative dielectric constant of the medium follows as:

$$\epsilon_r = 1 + \chi_E = 1 + \frac{Ne^2/m\epsilon_o}{(w_o^2 - w^2) - 2ibw}$$

and the index of refraction at frequency w for ϵ close to ϵ_o is:

$$n \equiv \sqrt{\epsilon_r} \cong 1 + \frac{Ne^2/2m\epsilon_o}{(w_o^2 - w^2) - 2ibw}$$

Now,

$$\vec{E} = \vec{E}_o e^{-i(wt-kz)}$$

in V/m. substituting $k = nw/c$ we get

$$\vec{E} = \vec{E}_o \exp\{-i[wt - (\mathrm{Re}\, n)\frac{w}{c}z]\} \exp[-\frac{w}{c}(\mathrm{Im}\, n)z]$$

which is a wave traveling along the $+z$ direction. Now, when we apply a magnetic field parallel to the $+z$ direction we split the spectral lines of the molecule due to the fact that the electron has a magnetic moment which is acted upon by the magnetic field. The splitting is proportional to the magnetic field and is known as the Zeeman effect. Lets assume, for simplicity, that our molecule can be idealized as a single electon atom. Now, the magnetic moment, $\vec{\mu}$, of any charge distribution is given as:

$$\vec{\mu} \equiv \frac{1}{2}\int (\vec{r} \times \vec{v})dq$$

in $C \cdot m^2/s$. Using the above formula we will find the magnetic moment of an electron in a circular orbit in terms of its angular momentum. This is:

$$\vec{\mu}_l = -\frac{e}{2m}\vec{L}$$

which shows that the magnetic moment and angular momentum vectors are always pointing in the opposite directions. Using the orbital g factor and the Bohr magneton we can rewrite the above as:

$$\vec{\mu}_l = -\frac{g_l \mu_o}{\hbar}\vec{L}$$

For atoms the orbital g factor is 1. Now, given an external magnetic field applied to the atom, the magnetic moment undergoes a Larmor precession about the magnetic field with angular frequency $w = \mu_o B/\hbar$. This causes an alteration of the electrons motion. The total energy of the electron now depends on the orientation of the magnetic moment and magnetic field because the magnetic field contributes energy to the electron:

$$E = -\vec{\mu}_l \cdot \vec{B}$$

where the energy is measured in Joules. This is not the total magnetic energy of the molecule since there is a contribution from the spin angular momentum, $\vec{S}$, of the electron:

$$\vec{\mu}_s = -\frac{g_s \mu_o}{\hbar} \vec{S}$$

where the g_s is the spin g factor. So, this leads to the actual total magnetic energy:

$$E = -(\vec{\mu}_l + \vec{\mu}_s) \cdot \vec{B}$$

Now, looking at the quantum mechanical description of the electron we note that the electron can be described by the wave function $\Psi_{n,l,m_l,m_s}(r, \theta, \phi)$ which is an eigenfunction of the Hamiltonian:

$$H_o = \frac{p^2}{2m} + V(r)$$

in Joules. V is the Coulomb potential energy of the electron. By looking at the wave equation we see that:

$$L = [l(l+1)]^{1/2}\hbar$$

$$S = [s(s+1)]^{1/2}\hbar$$

and in the z direction we have:

$$L_z = m_l \hbar$$

$$S_z = m_s \hbar$$

where m_s is $\pm\frac{1}{2}$. The energy of the state now depends on the quantum number n and l since there is no magnetic field. Now, if we apply a strong magnetic field such that the spin-orbit correction of the Hamiltonian is negligible. The Hamiltonian for the system becomes:

$$H \simeq H_o + H_z$$

where H_z is the Zeeman interaction Hamiltonian. The energy E is now calculated using the time-independent Schrodinger equation to yield:

$$E = E_o(n,l) + E_z = E_o(n,l) + \mu_o B(g_l m_l + g_s m_s)$$

but $g_l = 1$ and $g_s \cong 2$ so:

$$E = E_o(n,l) + E_z = E_o(n,l) + \mu_o B(m_l + 2m_s)$$

where E_o is the energy of the zero magnetic field Hamiltonian. All of the levels of the molecule are degenerate for $\vec{B} = 0$ and splitting occurs for $\vec{B}$. The extent of the splitting depends only on the sum $m_l + 2m_s$. So, the Zeeman effect causes the splitting of excited electron states in the magnetic field. Because of this splitting we have a difference in the absorption of right circularly polarized light and left circularly polarized light. This correlates directly to the fact that the splitting states have different values of L_z. Now, the LCP and RCP photons carry an angular momentum of $+\hbar$ and $-\hbar$, respectively, along the direction of propagation. By conservation of angular momentum, the absorption of either type of photon by the electrons causes a corresponding change in the system. Changing the angular momentum by ΔL_z changes its energy in the presence of a magnetic field by:

$$\Delta E = \Delta(-\vec{\mu}_l \cdot \vec{B}) = [\frac{e}{2m}(\Delta L_z)]B$$

in Joules. The difference in peak energies of the LCP and RCP photons causing the transitions is:

$$\Delta E = \hbar \Delta w = \frac{e}{2m}(2\hbar)B$$

so, the frequency separation of the two absorption peaks is:

$$\Delta w = \frac{e}{m}B$$

radians/sec.

Finally, referring back to our wave in the $+z$ direction we have:

$$\vec{E}(z,t) = E_o \sin(wt - kz)\hat{y}$$

where $k \equiv 2\pi/\lambda$ is the wave number. Now, we will assume that the wave is circularly polarized so:

$$\vec{E}(z,t) = E_o' \sin(wt - kz)\hat{x} + E_o' \sin(wt - kz \pm \frac{\pi}{2})\hat{y}$$

again measured in V/m. So, the x and y components of the wave are $\pi/2$ radians out of phase. For $+\pi/2$ we have RCP wave and $-\pi/2$ for LCP wave. Now, because of the Zeeman splitting we can assume that the medium is circularly birefringent which means that it has slightly different indices of refraction n_r and n_l for RCP and LCP, respectively. Since the indices of refraction are different for the LCP and RCP waves inside the medium then it follows that the speeds, wavelengths, and wave numbers will also be different inside the medium. For the LCP wave, the wave number, k_l, will be given by $k_l = 2\pi/\lambda_l = n_l k$, where k is the wave number of both components outside of the medium. Similarly for the RCP wave we have $k_r = n_r k$. If we assume that $n_r > n_l$ then the phase of the RCP component will be smaller than the phase of the LCP component inside of the medium, of length L, by an amount δ:

$$\delta = (n_r k)L - (n_l k)L = \frac{wL}{c}(n_r - n_l)$$

in radians. This means that the RCP component of the wave is an angle δ radians closer to the LCP component than it would have been in the absence of the medium. This resultant $\vec{E}$ is thus no longer oscillating along the y-direction, but has been rotated by an angle of $\theta = \delta/2$ radians. So, the Faraday rotation is given by:

$$\theta = \frac{wL}{2c}(n_r - n_l)$$

in radians.

Now, referring back to the quantum mechanical discussion we will approximate $n_r - n_l \equiv \Delta n \simeq (dn/dw)\Delta w$, where n is the real part of the index of refraction. So, the Faraday rotation is given as:

$$\theta = \frac{we}{2mc}(\frac{dn}{dw})BL$$

in radians. The derivative is evaluated at the angular frequency of the radiation. Notice that the Verdet constant has now been reduced to an expression of the refractive properties of the medium. The equation for the Verdet constant is thus:

$$V = \frac{we}{2mc}(\frac{dn}{dw})$$

Furthermore, we can express this in terms of the wavelength to get:

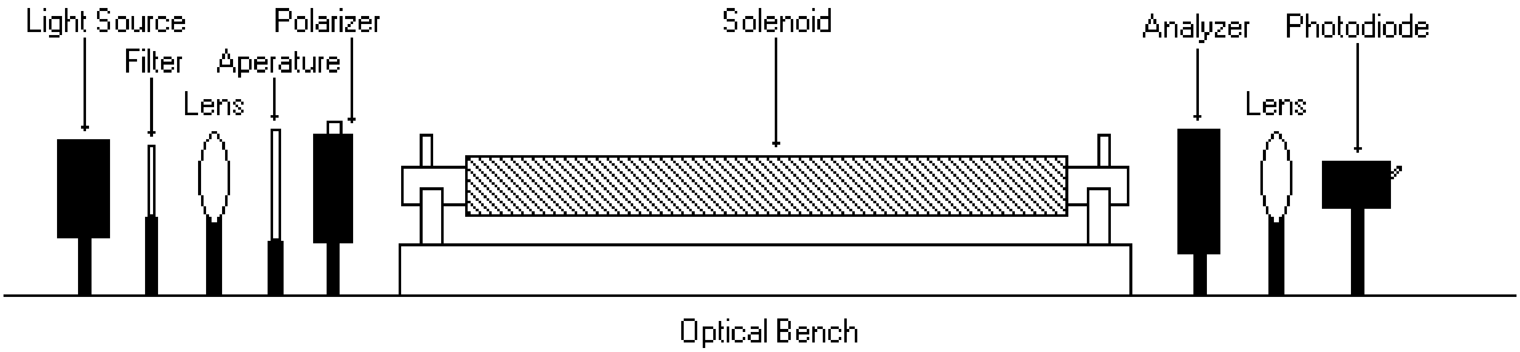
$$V = -\frac{e}{2mc}\lambda(\frac{dn}{d\lambda})$$

in radians. Knowing the frequency of the radiation and index of refraction one could use this to find e/m. This might be a good way to check ones results.

III. APPARATUS

Basically, the entire apparatus consisted of five parts:

1. Monochromatic Light Source
2. Polarizer
3. Solenoid Apparatus
4. Analyzer
5. Focusing Equipment



The monochromatic light source used was an incandescent light with a green filter attached. Even though the intensity of the incandescent source is only a fraction of that of a He-Ne laser, the incandescent proved to be more stable for longer periods of time. I decided to trade this stability for low intensity. By using an aperture and several focusing lenses I was able to compensate for the lack of intensity.

Another drawback to using the incandescent was the fact that the light was not polarized. The solution to this was to position a polarizer just before the solenoid apparatus. Before the polarizer, I placed a focusing arrangement. This arrangement consisted of a concave focusing lens and an aluminum aperture. The lens helped focus the light of the incandescent source while the aperture helped limit the cross-sectional area of the beam. This was crucial due to the fact that the incandescent had such a low intensity and to help limit any interference from rays reflecting or refracting of the inside of the glass cylinder of water, respectively.

This leads to the solenoid apparatus. The solenoid apparatus was essentially a two cylinder system. The outer cylinder was wrapped with 652 turns of 12 gauge wire. The inner cylinder contained the sample of distilled water which was to be analyzed. The space between the outer and inner cylinders was filled with regular tap water from a inlet and was allowed to flow at a slow pace to the drain through two projections from the outer cylinder. The reason water flowed through the system was to provide cooling for the solenoid. Without this cooling the water in the inner cylinder would have boiled at high currents. Not only could this have caused a potentially dangerous explosion, but the Faraday effect is dependent upon temperature. The dependence of the Faraday effect on temperature is related to the index of refraction and its inherent dependence upon temperature. The entire apparatus was mounted upon a wooden frame which straddled the optical bench. The solenoid was connected to a DC power supply which provided the current for the production of the magnetic field.

Next, we look at the analyzer setup. Basically, the analyzer consisted of a Newport Model 471 polarizer, photodiode, and Keithly 614 electrometer. The polarizer was of a vernier type which allowed for fine measurements of the angle of polarization. The photodiode was used to detect the intensity of the light after coming through the preceding apparatus. The photodiode detected the intensity of the light and this was measured via the output current of the photodiode. The gain of the photodiode was 999999 I measured this current using the electrometer with a scale set to micro amps.

IV. PROCEDURE

First, I will discuss the setup and alignment of the apparatus. I tried to position the equipment as shown in the figure. I began by placing the solenoid apparatus on the optical rail. I then proceeded to connect two pieces of surgical tubing to the outer cylinder. One section of tubing was attached to a nearby water outlet and the other was placed in a drain in the floor. This was to facilitate cooling of the solenoid. I wanted to prevent the solenoid from heating the inner cylinder of water since the index of refraction of water changes with temperature. Next, I placed the polarizer and analyzer at the opposite ends of the solenoid apparatus. I placed the incandescent light source in front of the polarizer. I then proceeded to place the photodiode behind the analyzer. I connected the photodiode to an electrometer. After connecting the photodiode, I placed a green color filter in front of the incandescent lamp. Next, I positioned an aluminum aperture directly before the polarizer. This was to help reduce any stray rays from interfering with the photodiode. Finally, I activated the incandescent lamp. After letting the lamp warm up, I turned on the photodiode and started the electrometer. I adjusted the lamp and aperture to obtain a small patch of green light on the front of the inner cylinder of the solenoid apparatus. I proceeded to check that the light was incident upon the polarizing film of the analyzer. With the aid of the electrometer, I adjusted the height of the photodiode until I obtained a maximum reading. The reading was quite low so I used two lenses to help focus the light. I placed one lens between the green filter and the aperture, and the other lens between the analyzer and photodiode.

After setting up the equipment, I measured the angles of polarization with maximum and minimum intensity via the electrometer with no current in the solenoid. For the maximum intensity I had -52^o and for the minimum intensity I had +38^o. This meant that the half-way point was at -7^o. All of my subsequent measurements used this as a reference point. The reason I used the half-way point as a reference point is because the observed intensity, I, is related to the maximum intensity, I_o, and angle of polarization, θ, by:

$$I = I_o \cos^2 \theta$$

Now, by differentiating with respect to θ this we get:

$$\frac{dI}{d\theta} = -2I_o \cos\theta \sin\theta$$

Differentiating with respect to θ again yields:

$$\frac{d^2I}{d\theta^2} = 2I_o^2(1 - 2\cos^2\theta)$$

So, setting the rhs of equation 3 to zero and solving θ for we get:

$$\theta = \pm\frac{1}{4}\pi$$

which means that at the half-way point the derivative of I with respect to θ is a maximum. This in turn means that the maximum amount of change occurs around the half-way point. So, this makes the half-way point one of the more sensitive angles with which to measure the rotation of the plane of polarization. Next, I recorded the observed current from the electrometer. This was an indication of the intensity at the half-way point. Throughout the experiment I would use this measurement to locate the new half-way point.

Finally, I was ready to begin observing the Faraday effect. I began by turning the DC power supply on and letting it warm up. I set the current to one amp and measured the change in the angle of polarization. I continued at one amp intervals up to 8 amps. With the available power supply I was unable to go above 8 amps. After recording the measurements, I calculated the magnetic field produced at each current level. I used the infinite solenoid approximation:

$$B = \mu_o n I$$

where B is the magnetic field strength measured in gauss, μ_o is the permeability of free space in $\frac{Gm}{A}$, n is the number of turns of wire per unit length, and I is the current in amps. The magnetic field strength ranged from 10.43 gauss at 1 amp to 83.44 gauss at 8 amps. Next, I proceeded to derive the Verdet constant of water by using the first equation for each magnetic field I used. With that the experiment was completed.

V. RESULTS

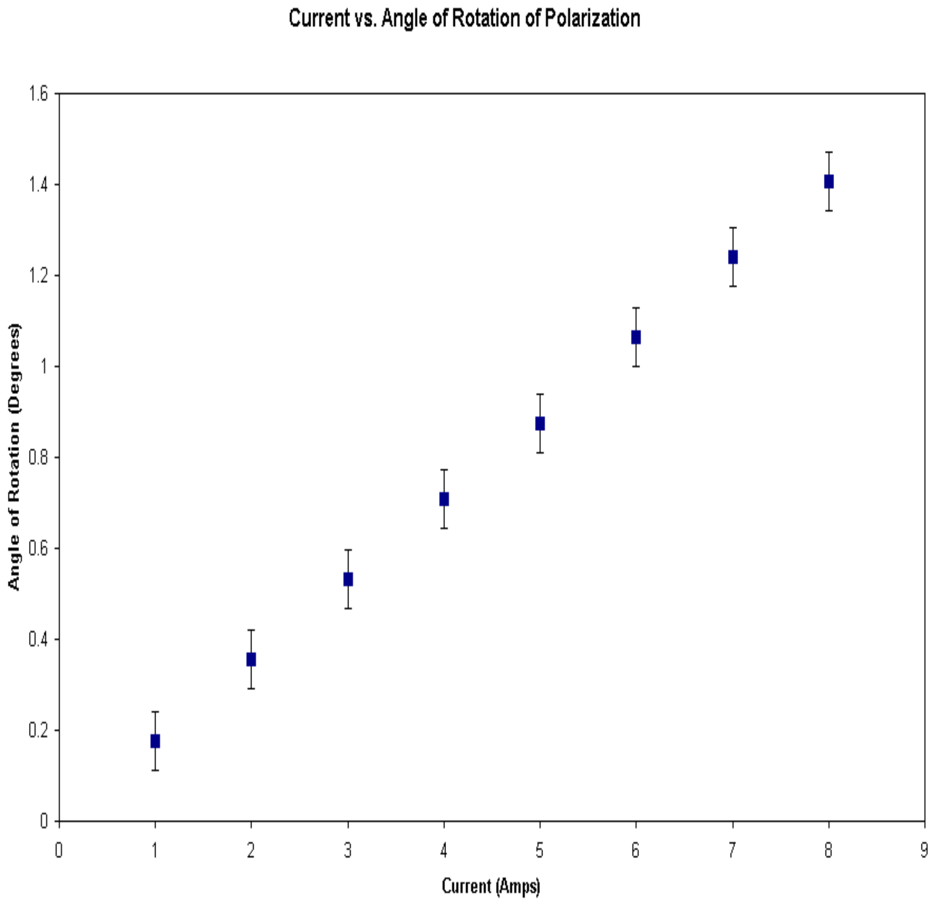
The following are the primary calculations and measurements I made during this experiment. First are my calculations of the magnetic field strength using the infinite solenoid approximation:

Current (Amps)	1	2	3	4	5	6	7	8
Magnetic Feild (Gauss)	10.43	20.86	31.29	41.72	52.15	62.58	73.01	83.44

As I applied the following currents they rotated the plane of polarization by the following angles:

Current (Amps)	1	2	3	4	5	6	7	8
Angle of Rotation (Degrees)	0.1773	0.3547	0.5320	0.7093	0.8740	1.064	1.241	1.406

The error for the current was ±0.001 A and the error in measuring the angle of rotation of the plane of polarization was ±0.005. Graphing the angle of rotation of the plane of polarization against the current shows the linear nature of the effect:



Next, I derived the Verdet constant for each magnetic field strength and angle measurement:

Magnetic Field (Gauss)	10.43	20.86	31.29	41.72	52.15	62.58	73.01	83.44
Angle of Rotation (Min)	10.64	21.28	31.92	42.56	52.44	63.84	74.48	84.36
Verdet Constant	0.01303	0.01302	0.01303	0.01303	0.01284	0.1302	0.01291	0.01299

Since the formula I used to calculate the Verdet constant used minutes I converted all of my angular measurements to minutes. Now by averaging over all of the values I arrived at a Verdet constant for water of 0.01299±0.00051 min/G cm. This means that my percent error was 3.93%.

VI. CONCLUSION

The experiment proceeded fairly well. I measured the Verdet constant for water and found it to be 0.01299±0.00051. The error involved was due to inaccuracies in measuring the DC power supply as well as in the analyzer. In future experiments a multimeter could be used to measure the current produced by the power supply. This would increase the accuracy without having to purchase a more accurate, therefore more expensive, DC power supply. To further eliminate any random errors, one could eliminate any stray light from interfering with the photodiode. Also, a more efficient photodiode could be used. The main problem I encountered with the experiment is the fact that the angle of rotation is extremely small. With a more accurate analyzer one could get a more accurate reading. Another avenue of development is to increase the strength of the magnetic field. Given the same path length of 78.5 cm, my calculations show that for a 10 degree rotation a magnetic field of 585.39 G would be needed. Another interesting area to explore to increase the overall accuracy would be to extend the path length through the water. Using mirrors at the ends of the solenoid one could increase the path length by reflecting the light beam back and forth through the solenoid apparatus. Since the faraday effect does not depend upon the direction of propagation of the light the paths would be additive resulting in a larger angle of rotation. Finally, one could use a HeNe laser to increase the intensity of the transmitted light. My experience has been that lasers tend to drift. The Verdet constant I arrived at differed from the CRC value of 0.1309 by 0.001 . Considering the propagation of errors, the value I obtained could be as far off as 0.0052 minutes. To further check ones result one could use the fact that the Verdet constant is dependent upon index of refraction and wavelength of the transmitted light to determine the quantity e/m. Overall, I am fairly confident in my measurements and believe that the arrived at Verdet constant is fairly accurate.

VII. REFERENCES

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