A major breakthrough in our understanding of local symmetries was obtained in 1974 by Becchi, Rouet and Stora who realized that for every local symmetry there corresponds a non-linear field dependent global symmetry which implies the local gauge invariance of the S matrix. The corresponding global symmetry transformation in fact leave $\mathcal{L}$ invariant; hence, the "broken" gauge WI is converted into an unbroken global WI — called the BRS identity.

In order to convert the above gauge WI into the BRS identity recall the $\phi\pi$ field equations. The Euler-Lagrange derivative of $\mathcal{L}$ is

$$\frac{\partial^E \mathcal{L}}{\partial \bar{C}_a(x)} = \int d^4y \, M_f^{ab}(x,y) C_b(y)$$

Hence when applied to Green functions we, as usual have "contact" terms due to the time ordering; thus

$$\boxed{\int d^4y\, M_f^{ab}(x,y)\, C_b(y)\, Z[J, J_\mu, \xi, \bar{\xi}] = -\,\xi_a(x)\, Z[J, J_\mu, \xi, \bar{\xi}]}$$

(Note: action principle yields field equations also i.e.

$$Z[J] = \int [d\varphi]\, e^{i\int dx\,(\mathcal{L}(\varphi) + J\varphi)}$$

let $\varphi \to \varphi + \delta\varphi$

$$0 = \int [d\varphi]\left[i\int dx\,\left(\mathcal{L}(\varphi + \delta\varphi, \partial_\mu\varphi + \partial_\mu\delta\varphi) - \mathcal{L}(\varphi, \partial_\mu\varphi) + J\delta\varphi\right]\, e^{i\int dx\,(\mathcal{L} + J\varphi)}$$

but $\delta\mathcal{L} = \dfrac{\partial^\varepsilon \mathcal{L}}{\partial\varphi}\,\delta\varphi$

So $$0 = \int [d\varphi]\left(i\int dx\left[\frac{\partial^\varepsilon \mathcal{L}}{\partial\varphi} + J\right]\delta\varphi\right) e^{i\int dx\,(\mathcal{L} + J\varphi)}$$

Then we obtain directly since $\delta\varphi$ is arbitrary

$$\frac{\partial^\epsilon \mathcal{L}}{\partial \varphi}(x)\, Z[J] = -J(x)\, Z[J]$$

the field equations).

Similarly the $\bar{c}$ equation of motion is

$$\boxed{\begin{aligned}\int d^4x\, d^4y\ \bar{c}_b(x)\, M_f^{ba}(x,y)\, \delta^4(z-y)\, Z[J, J_\mu, \xi, \bar{\xi}] \\ = -\bar{\xi}_a(z)\, Z[J, J_\mu, \xi, \bar{\xi}]\end{aligned}}$$

What BRS realized was that the RHS of the gauge WI involves the $\phi\pi$ equations of motion if we choose the arbitrary $\omega^a(x)$ to be the $\phi\pi$ ghost field $c^a(x)$

But $\omega^a(x)$ is a real function and $C^a(x)$ is an anti-commutating function thus we must let

$$\omega^a(x) = \delta\lambda\, C^a(x)$$ where

$\delta\lambda$ is an infinitesimal constant anti-commuting number

$$(\delta\lambda)^2 = 0, \quad \delta\lambda\, C^a = -C^a\delta\lambda$$
$$\delta\lambda\, \bar{C}^a = -\bar{C}^a\delta\lambda$$

then the local Gauge WI becomes

$$\int d^4x \left[J_\alpha \delta_Q(\delta\lambda C)\varphi^\alpha + J^a_\mu \delta_Q(\delta\lambda C) A^{a\mu} \right] Z$$

$$= \int dx\, dy \left[\delta\lambda\, \xi_a(x) M_f^{ab}(x,y) C^b(y) - \bar{C}_a(x) \left(\delta_Q(\delta\lambda C) M_f^{ab}(x,y) \right) \xi_b(y) \right] Z$$

So the first thing to note is that

$$\boxed{\begin{aligned}&\int dy\, f_a(x)\, M_f^{ab}(x,y)\, C^b(y)\, Z[J, J_\mu, \xi, \bar{\xi}] \\ &\qquad = -f_a(x)\, \xi_a(x)\, Z[J, J_\mu, \xi, \bar{\xi}]\end{aligned}}$$

due to the C^a-field equation.

The next term on the RHS is a bit more complicate to evaluate, it involves

$$\int dy\, \delta_Q(\delta\lambda C)\, M_f^{ab}(x,y)\, C_b(y)$$

recall $M_f^{ab}(x,y) \equiv \dfrac{\delta_Q(\omega) f_a(x)}{\delta \omega^b(y)}$

and $\delta_Q(\omega')\, M_f^{ab}(x,y) = \displaystyle\int dz \frac{\delta_Q(\omega')\,\delta_Q(\omega) f_a(x)}{\delta\omega'^d(z)\,\delta\omega^b(y)}\, \omega'^d(z)$

So for $\omega' = \delta\lambda C$ we have

$$\delta_Q(\delta\lambda C)\, M_f^{ab}(x,y) = \int d^4z\, \frac{\delta_Q(\omega')\,\delta_Q(\omega)\, f_a(x)}{\delta\omega'^{d}(z)\,\delta\omega^{b}(y)}\, \delta\lambda C^{d}(z)$$

Hence

$$\int d^4y\, \delta_Q(\delta\lambda C)\, M_f^{ab}(x,y)\, C_b(y)$$

$$= \int d^4y\, d^4z\, \delta\lambda\, \frac{\delta_Q(\omega')\,\delta_Q(\omega)\, f_a(x)}{\delta\omega'^{d}(z)\,\delta\omega^{b}(y)}\, C^{d}(z)\, C^{b}(y)$$

Since $C^d(z)\, C^b(y) = -C^b(y)\, C^d(z)$

because the ϕ-Π fields are anti-commuting we find the commutator of gauge transformations,

$$= \int dy\, dz\, \delta\lambda \left[\frac{\delta_Q(\omega')\,\delta_Q(\omega)\, f_a(x)}{\delta\omega'^{d}(z)\,\delta\omega^{b}(y)} - \frac{\delta_Q(\omega')\,\delta_Q(\omega)\, f_a(x)}{\delta\omega'^{b}(y)\,\delta\omega^{d}(z)} \right]$$

$$\times\, C^d(z)\, C^b(y)$$

$$= \int dy\, dz\, \delta\lambda \left[\frac{\delta_Q(\omega')\, \delta_Q(\omega)\, f_a(x)}{\delta\omega'^d(z)\, \delta\omega^b(y)} - \frac{\delta_Q(\omega)\, \delta_Q(\omega')\, f_a(x)}{\delta\omega^b(y)\, \delta\omega'^d(z)} \right]$$

$$C^d(z)\, C^b(y)$$

$$= \int dy\, dz\, \delta\lambda\, \frac{\delta^2}{\delta\omega'^d(z)\, \delta\omega^b(y)} \left[\delta_Q(\omega'), \delta_Q(\omega)\right] f_a(x) \times$$

$$\times C^d(z)\, C^b(y)$$

but $\left[\delta_Q(\omega'), \delta_Q(\omega)\right] = \delta_Q(\ \omega'')$

(p. 514 -) where $\omega''^a = f_{abc}\, \omega^{b'}\, \omega^c$

Thus

$$\int d^4y\, \delta_Q(\delta\lambda c)\, M_f^{ab}(x,y)\, C_b(y)$$

$$= \int dy\, dz\, \delta\lambda\, \frac{\delta^2}{\delta\omega'^d(z)\, \delta\omega^b(y)}\, \delta_Q\left(f_{abc}\, \omega'^b\, \omega^c\right) f_a(x)$$

$$C^d(z)\, C^b(y)$$

$$= \int dy\, dz\, \delta\lambda \frac{\delta^2}{\delta\omega'^d(z)\,\delta\omega^b(y)} \int dt\, M_f^{ae}(x,t)$$

$$\left(f_{egh}\, \omega'^g(t)\, \omega^h(t)\right) c^d(z)\, c^b(y)$$

$$= \int dy\, dz\, dt\, \delta\lambda\, M_f^{ae}(x,t)\, f_{egh}\, \delta^{dg}\, \delta^4(t-z)$$

$$\delta^{hb}\, \delta^4(t-y)\, c^d(z)\, c^b(y)$$

$$= \int dt\, \delta\lambda\, M_f^{ae}(x,t)\, f_{edb}\, c^d(t)\, c^b(t)$$

$$\boxed{\begin{aligned} &= \int d^4y\, \delta\lambda\, M_f^{ab}(x,y)\, f_{bcd}\, c^c(y)\, c^d(y) \\ &= \int d^4y\, \delta_Q(\delta\lambda c)\, M_f^{ab}(x,y)\, c_b(y) \end{aligned}}$$

So the second term is of the form

$$
\boxed{
\begin{aligned}
&-\int dx\,dy\ \bar{c}_a(x)\left(\delta_Q(\delta\lambda c)\,M_f^{ab}(x,y)\right)c_b(y)\,Z \\
&= -\int dx\,dy\ \bar{c}_a(x)\,\delta\lambda\,M_f^{ab}(x,y)\,f_{bcd}\,c^c(y)\,c^d(y)\,Z \\
&= +\delta\lambda\int dx\,dy\ \bar{c}_a(x)\,M_f^{ab}(x,y)\,f_{bcd}\,c^c(y)\,c^d(y)\,Z \\
&= +\delta\lambda\int dx\ \left(-\bar{\xi}_b(x)\right)f_{bcd}\,c^c(x)\,c^d(x)\,Z \\
&= -\delta\lambda\int d^4x\ \bar{\xi}_a(x)\,f_{abc}\,c^b(x)\,c^c(x)\,Z[J,J_\mu,\xi,\bar{\xi}]
\end{aligned}
}
$$

by using the field equation on page -589- for $\bar{c}_b$.

Putting these two expressions into the RHS of the gauge WI we obtain

$$\int d^4x \left[J_\alpha \delta_Q(\delta\lambda c)\varphi^\alpha + J_\mu^a \delta_Q(\delta\lambda c) A^{a\mu} \right] Z[J, J_\mu, \xi, \bar{\xi}]$$

$$= -\delta\lambda \int d^4x\, f_a(x)\, \xi_a(x)\, Z[J, J_\mu, \xi, \bar{\xi}]$$

$$-\delta\lambda \int d^4x\, \bar{\xi}_a(x) f_{abc} C^b(x) C^c(x)\, Z[J, J_\mu, \xi, \bar{\xi}]$$

Thus all terms involve sources, bringing them to the RHS we have the BRS identity

$$\boxed{\int d^4x \left[J_\alpha \delta_{BRS} \varphi^\alpha + J_\mu^a \delta_{BRS} A^{a\mu} + \xi_a \delta_{BRS} \bar{C}_a - \bar{\xi}_a \delta_{BRS} C_a \right] \times Z[J, J_\mu, \xi, \bar{\xi}]}$$

where the BRS transformation of the φ, A_μ fields is just the gauge variation of these fields with the gauge parameter ω^a replaced with the ϕ-π field $C^a(x)$

$$\delta_{BRS}\,\varphi^\alpha(x) \equiv -i C^a(x) [T^a]_{\alpha\beta}\, \varphi^\beta(x)$$

$$\delta_{BRS}\, A^a_\mu(x) \equiv \partial_\mu C^a(x) + f_{abc} C^b(x) A^c_\mu(x)$$

and, as indicated by the source term for the ϕ-π fields, their BRS transformations are defined as

$$\delta_{BRS}\, C^a(x) \equiv - f_{abc} C^b(x) C^c(x)$$

$$\delta_{BRS}\, \bar{C}^a(x) \equiv f_a[A_\mu, \varphi](x) .$$

The BRS identity describes the invariance of Z under an anti-commuting global symmetry associated with gauge transformations

That is we could define the above BRS transformations and apply Noether's theorem and the quantum action principle to them to find (see p.-441-)

$$\partial_\mu J^\mu_{BRS}(x)\, Z\{J, J_\mu, \xi, \bar{\xi}\}$$

$$= \Big[J_\alpha \delta_{BRS} \varphi^\alpha + J^a_\mu \delta_{BRS} A^{a\mu} - \bar{\xi}_a \delta_{BRS} C^a + \delta_{BRS} \bar{C}^a \xi_a \Big] Z$$

(Notice minus since we have fermions $\delta\lambda$ out through $\bar{\xi}$ i.e. $\delta\lambda\bar{\xi} = -\bar{\xi}\delta\lambda$)

$$+ \delta_{BRS} \mathcal{L}(x)\, Z$$

where

$$J^\mu_{BRS}(x) \equiv \Big[\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^\alpha} \delta_{BRS} \varphi^\alpha + \frac{\partial \mathcal{L}}{\partial \partial_\mu A^a_\nu} \delta_{BRS} A^a_\nu$$

$$+ \delta_{BRS} \bar{C}^a \frac{\partial \mathcal{L}}{\partial \partial_\mu \bar{C}^a} - \frac{\partial \mathcal{L}}{\partial \partial_\mu C^a} \delta_{BRS} C^a \Big]$$

minus since δ_{BRS} anti commutes!

Let's check the invariance of $\mathcal{L}$

Recall $\mathcal{L} = \mathcal{L}_{inv}(\varphi, A_\mu) + \mathcal{L}_f + \mathcal{L}_{\phi\pi}$

$\mathcal{L}_{inv} = \mathcal{L}(\varphi, D_\mu\varphi) - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu}$

$\mathcal{L}_f = -\frac{1}{2} f_a f_a$

$\mathcal{L}_{\phi\pi} = \int d^4y \; \bar{c}_a(x) M_f^{ab}(x,y) c_b(y)$

with
$$M_f^{ab}(x,y) \equiv \frac{\delta_Q(\omega) f_a(x)}{\delta \omega^b(y)}$$

Now
$$\delta_{BRS} \mathcal{L}_{inv} = 0 \quad \text{since } \delta_{BRS} \text{ is}$$
just a gauge transformation with parameter c^a and $\mathcal{L}_{inv}$ is gauge invariant. Further

$$\delta_{BRS} \mathcal{L}_f = - f_a \, \delta_{BRS} f_a$$
$$= - f_a \frac{\delta_Q(\delta\lambda c) f_a}{\delta\lambda} = - f_a(x) \int dy \, M_f^{ab}(x,y) \times c^b(y)$$

since f_a involves A_μ^a, φ^α only.

finally

$$\delta_{BRS}\mathcal{L}_{\phi\pi} = \int d^4y\, \delta_{BRS}\bar{c}_a(x)\, M_f^{ab}(x,y)\, c_b(y)$$

$$-\int d^4y\, \bar{c}_a(x)\, \delta_{BRS}\left(M_f^{ab}(x,y)\, c_b(y)\right)$$

The minus sign since δ_{BRS} anti-commutes

$$\delta_{BRS}\mathcal{L}_{\phi\pi} = \int d^4y\; f_a(x)\, M_f^{ab}(x,y)\, c_b(y)$$

$$-\int d^4y\, \bar{c}_a(x)\left(\delta_{BRS} M_f^{ab}(x,y)\right) c_b(y)$$

$$-\int d^4y\, \bar{c}_a(x)\, M_f^{ab}(x,y)\left[-f_{bcd}\, c^c(y)\, c^d(y)\right]$$

Once again we know from pages -592- to -594-

$$\int d^4y \left(\delta_{BRS} M_f^{ab}(x,y)\right) c_b(y) = \int d^4y\, M_f^{ab}(x,y)\, f_{bcd}\, c^c(y)\, c^d(y)$$

Thus

$$\delta_{BRS}\mathcal{L}_{\phi\pi} = +\int d^4y\, f_a(x)\, M_f^{ab}(x,y)\, c_b(y)$$

$$-\cancel{\int d^4y\, \bar{c}_a(x)\, M_f^{ab}(x,y)\, f_{bcd}\, c^c(y)\, c^d(y)}$$

$$+\cancel{\int d^4y\, \bar{c}_a(x)\, M_f^{ab}(x,y)\, f_{bcd}\, c^c(y)\, c^d(y)}$$

Thus

$$\boxed{\delta_{BRS}(\mathcal{L}_f + \mathcal{L}_{\phi\pi}) = 0}$$

Hence $\boxed{\boxed{\delta_{BRS}\mathcal{L} = 0}}$

The BRS variations are a global symmetry of $\mathcal{L}$.

Hence integrating over x we obtain the action principle which with $\delta_{BRS}\mathcal{L} = 0$ secures the BRS identity (i.e. the global WI for BRS transformations)

$$\boxed{\begin{aligned} 0 = \int d^4x \Big[& J_\alpha \delta_{BRS}\varphi^\alpha + J_\mu^a \delta_{BRS} A^{a\mu} \\ & - \bar{\xi}_a \delta_{BRS} c^a + \delta_{BRS}\bar{c}^a \xi_a \Big] Z[J, J_\mu, \xi, \bar{\xi}] \end{aligned}}$$

<u>Technical Note:</u>

The $\mathcal{L}_{\phi\pi} = \int d^4y\, \bar{C}_a(x) M_f^{ab}(x,y) C_b(y)$ will in general have second order derivatives acting on C_b. Since we assumed $\mathcal{L}$ involved first order derivatives when we derived Noether's theorem we must perform an integration by parts — i.e. $\mathcal{L}_{\phi\pi}$ is first order in derivatives plus a total divergence term, which we ignore. J^μ_{BRS} is calculated with the first order derivative piece only. Hence

$$\delta_{BRS}(\mathcal{L}'_{\phi\pi} + \mathcal{L}_f) = \partial_\mu(-B^\mu)$$

$\mathcal{L}'_{\phi\pi}$ = first order terms, $\partial_\mu B^\mu$ is the δ_{BRS} variation of these total derivative terms we dropped.

BRS then went on to ask what is the most general, dimension 4, Lagrangian invariant under BRS transformations. They proved in fact that (for simple groups) it is just the Φ-Π gauge Lagrangian (for $U(1)$ factors we can add $m^2(\frac{1}{2}A_\mu A^\mu + \bar{C}C)$ and still be BRS invariant — this is just adding a mass to (photon) $U(1)$ gauge fields — we already know we can do this w/o destroying gauge invariance of S-matrix).
Further they showed that the BRS invariance of the S-matrix is equivalent to the gauge invariance, etc. Hence local Gauge invariance is equivalent to

global BRS invariance. Since the BRS WI are simpler it is easiest to work with them and all present day theories are formulated in terms of BRS invariance; including gravity and strings.

Example: QCD: consider an SU(3) gauge theory with gauge fields $G^a_\mu(x)$; called gluons $a = 1,2,3,\ldots,8$, and Dirac fermion quark fields $q^\alpha_i(x)$ where $\alpha = 1,2,3$ (color), and there are 6 types of quarks (u, d, c, s, t, b) labelled by index i (flavor) which we surpress — or better consider just one flavor say up.

The quarks are in the fundamental representation thus

$$\mathcal{U}^{-1}(\omega)\, q^{\alpha}(x)\, \mathcal{U}(\omega) = U(\omega)_{\alpha\beta}\, q^{\beta}(x)$$

$$= \left(e^{-ig\,\omega^a(x) T^a}\right)_{\alpha\beta} q^{\beta}(x)$$

where $(T^a)_{\alpha\beta} \equiv \frac{1}{2}(\lambda^a)_{\alpha\beta}$

with λ^a, $a=1,\ldots,8$ the 3×3 Gell-Mann λ-matrices

$$\lambda^1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad \lambda^2 = \begin{bmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad \lambda^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\lambda^4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\quad \lambda^5 = \begin{bmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ +i & 0 & 0 \end{bmatrix},\quad \lambda^6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\lambda^7 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{bmatrix};\quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

with

$$[T^a, T^b] = i f_{abc} T^c$$ where f_{abc} are completely anti-symmetric the SU(3) structure constants

$$f_{123} = +1$$

$$f_{147} = f_{246} = f_{257} = f_{345} = +\tfrac{1}{2}$$

$$f_{156} = f_{367} = -\tfrac{1}{2}$$

$$f_{458} = \tfrac{1}{2}\sqrt{3} = f_{678}$$

all other f's (not related by a permutation) are zero. Further we have factored out a strong coupling constant or unit of color charge explicitly; g.

So infinitesimally we have

$$[Q^a, q^\alpha(x)] = -\left(\frac{\lambda^a}{2}\right)_{\alpha\beta} q^\beta(x) \equiv -i\,\delta_Q^a q^\alpha$$

So for local transformations

$$\delta_Q(\omega)\, q^\alpha(x) = -i\omega^a(x)\left(\frac{\lambda^a}{2}\right)_{\alpha\beta} q^\beta(x)$$

The global SU(3) invariant Lagrangian is given by

$$\boxed{\mathcal{L}(q, \partial_\mu q) = \bar{q}^\alpha i\overrightarrow{\partial\!\!\!/} q^\alpha - m\bar{q}^\alpha q^\alpha}$$

(take derivative to right only for convenience

i.e. q^α is a 3 of SU(3)

$\bar{q}^\alpha$ is a $\bar{3}$ of SU(3)

$$\delta_Q(\omega)\,\bar{q}^\alpha(x) = \bar{q}^\beta(x)\left(\frac{\lambda^a}{2}\right)_{\beta\alpha}\left(+i\omega^a(x)\right)$$

The SU(3) Yang-Mills fields transform as

$$\delta_Q(\omega)\, G^a_\mu(x) = \partial_\mu \omega^a + g f_{abc}\, \omega^b(x)\, G^c_\mu(x)$$

Hence the SU(3) fundamental rep. covariant derivative is

$$(D_\mu q)^\alpha = D_\mu^{\alpha\beta} q_\beta$$
$$= \partial_\mu q_\alpha + i g G^a_\mu \left(\tfrac{\lambda^a}{2}\right)_{\alpha\beta} q^\beta$$

and the $\mathcal{L}(q, \partial_\mu q)$ can be made locally SU(3) invariant by a $\partial_\mu \to D_\mu$ substitution

$$\boxed{\delta_Q(\omega)\mathcal{L}(q, D_\mu q) = 0.}$$

$$\mathcal{L}(q, D_\mu q) = \bar{q}^\alpha i \not{D}^{\alpha\beta} q^\beta - m \bar{q}^\alpha q^\alpha$$
$$= \bar{q}\, i(\not{\partial} + g \not{G})\, q - m \bar{q} q$$

where

$$G_\mu \equiv i\left(\frac{\lambda^a}{2}\right) G^a_\mu .$$

The covariant field strength tensor is given by

$$G^a_{\mu\nu} \equiv \partial_\mu G^a_\nu - \partial_\nu G^a_\mu - g f_{abc} G^b_\mu G^c_\nu$$

Hence the SU(3) invariant Lagrangian is

$$\boxed{\mathcal{L}_{inv} \equiv \bar{q}[i\not\partial + ig\not{G} - m]q - \tfrac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}}$$

As a gauge fixing term we can take the Stückelberg type gauge

$$\boxed{f_a \equiv \frac{1}{\alpha} \partial^\mu G^a_\mu} \qquad \alpha = \text{arbitrary real parameter}$$

Then $\delta_Q(\omega) f_a = \frac{1}{\alpha} \partial^\mu (\partial_\mu \omega^a + g f_{abc} \omega^b G^c_\mu)$

So $$\boxed{M^{ab}_f(x,y) = \frac{1}{\alpha} \partial^\mu_x [\partial^x_\mu \delta^{ab} + g f_{abc} G^c_\mu(x)] \delta^4(x-y)}$$

Hence

$$\mathcal{L}_f = -\frac{1}{2\alpha}(\partial^\mu G^a_\mu)(\partial^\nu G^a_\nu) = -\frac{1}{2\alpha}(\partial^\mu G^a_\mu)^2$$

and

$$\mathcal{L}_{\phi\pi} = \int d^4y\, \bar{c}_a(x) M_f^{ab}(x,y) c_b(y).$$

$$\mathcal{L}_{\phi\pi} = -\frac{1}{\alpha}\partial^\mu \bar{c}^a(x)\partial_\mu c^a(x)$$
$$-\frac{1}{\alpha}\partial^\mu \bar{c}^a(x)\left[g f_{abc} c^b(x) G^c_\mu(x)\right]$$
$$+\underbrace{\frac{1}{\alpha}\partial^\mu(\bar{c}^a \partial_\mu c^a)}_{\text{drop total divergence.}}$$

The generating functional is given by

$$Z[\eta,\bar{\eta},J_\mu,\zeta,\bar{\zeta}] \equiv \int [dq^\alpha][d\bar{q}_\alpha][dG^a_\mu][dC^a][d\bar{C}^a] \times$$

$$\times e^{i\int d^4x\left[\mathcal{L} + \bar{\eta}^\alpha q^\alpha + \bar{q}^\alpha \eta^\alpha + J^a_\mu G^{a\mu} + \bar{\zeta}^a C^a + \bar{C}^a \zeta^a\right]}$$

with

$$\mathcal{L} = \mathcal{L}_{inv} + \mathcal{L}_f + \mathcal{L}_{\phi\pi}$$

$$\mathcal{L}_{inv} = \bar{q}\,[i\not{\partial} - m]\,q + ig\,\bar{q}\,\not{G}\,q - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu}$$

$$\mathcal{L}_f = \frac{-1}{2\alpha}\left(\partial^\mu G^a_\mu\right)^2$$

$$\mathcal{L}_{\phi\pi} = -\cancel{\frac{1}{\alpha}}\,\partial_\mu \bar{C}\,\partial^\mu C - \cancel{\frac{g}{\alpha}}\,\partial^\mu \bar{C}^a f_{abc} C^b G^c_\mu$$

we can develop a perturbation expansion in g with $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$

$$\mathcal{L}_0 = \bar{q}(i\not{\partial} - m)q - \frac{1}{4}\left[\partial_\mu G_\nu^a - \partial_\nu G_\mu^a\right]^2 - \frac{1}{2\alpha}\left(\partial^\mu G_\mu^a\right)^2$$
$$- \cancel{\tfrac{1}{\alpha}}\,\partial_\mu \bar{C} \partial^\mu C$$
α OK

$$\mathcal{L}_I = ig\bar{q}\not{G}q + \frac{1}{2}g f_{abc}\left[\partial_\mu G_\nu^a - \partial_\nu G_\mu^a\right] G^{b\mu} G^{c\nu}$$
$$- \frac{1}{4}g^2 f_{abc} f_{ade}\, G_\mu^b G_\nu^c G^{d\mu} G^{e\nu}$$
$$- \frac{g}{\cancel{\alpha}}\,\partial^\mu \bar{C}^a f_{abc} C^b G_\mu^c$$
α OK

The Green function are given by

$$\langle 0 | T\, q^{\alpha_1}(x_1) \cdots q^{\alpha_m}(x_m)\, \bar{q}^{\beta_1}(y_1) \cdots \bar{q}^{\beta_n}(y_n) \times$$
$$\times\, G_{\mu_1}^{a_1}(z_1) \cdots G_{\mu_\ell}^{a_\ell}(z_\ell) | 0 \rangle$$

$$= \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 \bar{p}_1}{(2\pi)^4} \cdots \frac{d^4 q_1}{(2\pi)^4} \cdots \frac{d^4 q_\ell}{(2\pi)^4}\, e^{-ip_i x_i}\, e^{+i\bar{p}_j y_j}$$
$$e^{-iq_k z_k} \sum_{\Gamma \in G^{(m,n,\ell)}}' \alpha(\Gamma)\, \delta_\Gamma \int \frac{d^4 h_1}{(2\pi)^4} \cdots \frac{d^4 h_{m(\Gamma)}}{(2\pi)^4}\, I_\Gamma$$

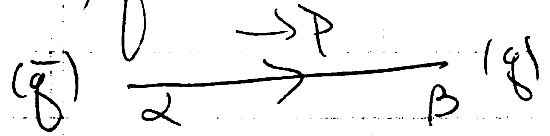
where $\mathcal{L}(\Gamma)$ = symmetry #

$\delta_\Gamma = (2\pi)^4\delta^4(p+\dots+q+\dots-\bar{p}\dots)$ for each connected factor of Γ. $m(\Gamma)$ = # of loops

$I_\Gamma = I_\Gamma(p,\bar{p},q,k)$ Feynman integrand made from graphical rules:

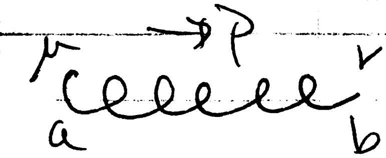
A) Propagators in Γ — Factor in I_Γ

1) quark lines



$$\left(\frac{\delta^{\alpha\beta}\, i}{\not{p}-m}\right)$$

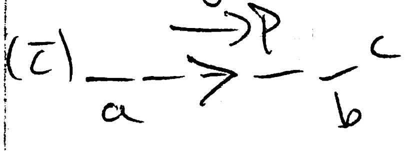
2) Gluon lines



$$\left(\frac{-i\delta^{ab}}{p^2}\left(g^{\mu\nu}-\frac{p^\mu p^\nu}{p^2}\right) - \frac{i\alpha\delta^{ab}}{p^2}\frac{p^\mu p^\nu}{p^2}\right)$$

$$=\left(\frac{-i\delta^{ab}}{p^2}g^{\mu\nu} - \frac{i\delta^{ab}}{p^2}(\alpha-1)\frac{p^\mu p^\nu}{p^2}\right)$$

A.) 3) ϕ-π ghost lines

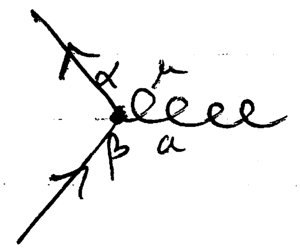


$$\frac{-i\delta^{ab}\alpha}{p^2}$$

B) Vertices

Vertex in Γ | Vertex factor in $i\Gamma$

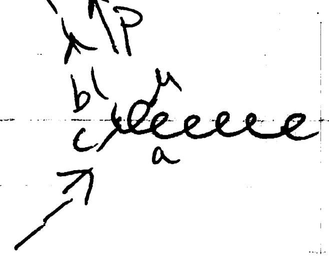
1)



$$g\left(i\frac{\lambda^a}{2}\right)_{\alpha\beta}\gamma^\mu$$

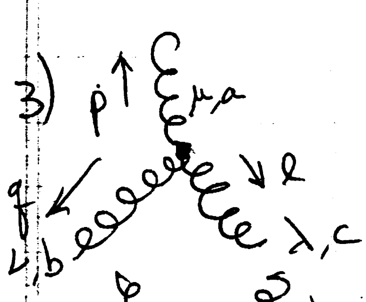
(Dirac indices surpressed)

2)



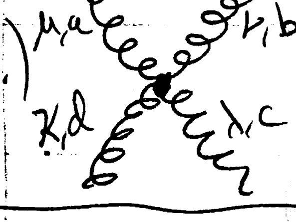
$$+\frac{g}{\cancel{\alpha}}p^\mu f_{bac}$$

3)



$$+g f_{abc}\left[(p-q)^\lambda g^{\mu\nu} + (q-l)^\mu g^{\nu\lambda} + (l-p)^\nu g^{\mu\lambda}\right]$$

4)



$$-ig^2\left[f_{eab}f_{ecd}(g_{\mu\lambda}g_{\nu\kappa}-g_{\mu\kappa}g_{\nu\lambda}) + f_{eac}f_{edb}(g_{\mu\kappa}g_{\lambda\nu}-g_{\mu\nu}g_{\lambda\kappa}) + f_{ead}f_{ebc}(g_{\mu\nu}g_{\lambda\kappa}-g_{\mu\lambda}g_{\nu\kappa})\right]$$

(clearly messy)

The BRS invariance should be checked explicitly; The BRS transformations are

$$\left(\delta_{BRS}\,\bar{q}^{\alpha}(x) = +\,\bar{q}^{\beta}(x)\,C^a(x)\left(\frac{i\lambda^a}{2}\right)_{\beta\alpha}\right)$$

$$\delta_{BRS}\, q^{\alpha}(x) = -i\left(\frac{\lambda^a}{2}\right)_{\alpha\beta} C^a(x)\, q^{\beta}(x)$$

$$\delta_{BRS}\, G^a_{\mu}(x) = \partial_{\mu} C^a(x) + g\, f_{abc}\, C^b(x)\, G^c_{\mu}(x)$$

$$\delta_{BRS}\, C^a(x) = -f_{abc}\, C^b(x)\, C^c(x)$$

$$\delta_{BRS}\, \bar{C}_a(x) = \frac{1}{\alpha}\partial^{\mu} G^a_{\mu}(x)$$

<u>Find</u> the current and check that it is conserved — <u>Note</u>: you must add to it the correction arising from the fact that $\delta_{BRS}\mathcal{L}$ = total divergence.

Check this $\delta_{BRS}\mathcal{L}$ = total divergence

End