Finally we can ask about the Schwinger action principle for local symmetries, thus we will derive the gauge Ward Identities. As usual we perform a change of integration variables corresponding to a gauge transformation (note: the ϕ-π fields do not transform under gauge transformations)

$$Z[J, J_\mu, \xi, \bar{\xi}] = \int [dA^{a\prime}_\mu][d\varphi^{\alpha\prime}][dC_a][d\bar{C}_a] \times$$
$$\times e^{i\int dx\left[\mathcal{L}(\varphi', A'_\mu, C, \bar{C}) + J_\alpha \varphi^{\alpha\prime} + J^a_\mu A^{a\mu\prime} + \bar{\xi}^a C_a + \bar{C}_a \xi^a\right]}$$

Now let
$$A^{a\prime}_\mu = A^a_\mu + \partial_\mu \omega^a(x) + f_{abc}\,\omega^b(x) A^c_\mu(x)$$
$$= A^a_\mu + \delta_Q(\omega) A^a_\mu$$
$$\varphi^{\alpha\prime} = \varphi^\alpha - i\omega^a(x)(T^a)_{\alpha\beta}\,\varphi^\beta(x)$$
$$= \varphi^\alpha + \delta_Q(\omega)\varphi^\alpha$$

and recall that $[dA^{a\prime}_\mu] = [dA^a_\mu]$; $[d\varphi^{\alpha\prime}] = [d\varphi^\alpha]$ that is we still integrate over all field configurations and

$$\mathcal{L}(\varphi', A'_\mu, c, \bar{c}) = \mathcal{L}(\varphi, A_\mu, c, \bar{c}) + \delta_Q(\omega)\mathcal{L}(\varphi, A_\mu, c, \bar{c})$$

So

$$Z[J, J_\mu, \xi, \bar{\xi}] = \int [dA^a_\mu][d\varphi^\alpha][dc_a][d\bar{c}_a] \times$$

$$\times\, e^{i\int dx\left[\mathcal{L}(\varphi, A_\mu, c, \bar{c}) + J_\alpha \varphi^\alpha + J^a_\mu A^{a\mu} + \bar{\xi}^a c_a + \bar{c}_a \xi^a\right]}$$

$$\times\, e^{i\int dx\left[\delta_Q(\omega)\mathcal{L}(\varphi, A_\mu, c, \bar{c}) + J_\alpha \delta_Q(\omega)\varphi^\alpha + J^a_\mu \delta_Q(\omega) A^{a\mu}\right]}$$

Since ω^a is infinitesimal we expand the exponential as usual to find

$$Z[J, J_\mu, \xi, \bar{\xi}] = Z[J, J_\mu, \xi, \bar{\xi}]$$

$$+ \int [dA^a_\mu][d\varphi^\alpha][dc_a][d\bar{c}_a] \times$$

$$\times \left\{ i\int dx\left[\delta_Q(\omega)\mathcal{L}(\varphi, A_\mu, c, \bar{c}) + J_\alpha \delta_Q(\omega)\varphi^\alpha + J^a_\mu \delta_Q(\omega) A^{a\mu}\right]\right\}$$

$$\times\, e^{i\int dx\left[\mathcal{L} + J\varphi + J_\mu A^\mu + \bar{\xi}c + \bar{c}\xi\right]}$$

Then we find the gauge Ward identity

$$\boxed{0 = \int d^4x \left[\delta_G(\omega)\mathcal{L}(\varphi, A_\mu, c, \bar{c}) + J_\alpha \delta_G(\omega)\varphi^\alpha + J_\mu^a \delta_G(\omega) A^{a\mu} \right] Z[J, J_\mu, \xi, \bar{\xi}]}$$

Where in the above notation it's understood that $\delta_G(\omega)\mathcal{L}$, $\delta_G(\omega)\varphi^\alpha$, $\delta_G(\omega)A^{a\mu}$ are to be written in terms of derivatives with respect to the appropriate sources $J, J_\mu, \xi, \bar{\xi}$.

First note that this WI or action principle is local that is $\omega^a = \omega^a(x)$ is an arbitrary function of x^μ thus we can functionally differentiate w.r.t. it and obtain an local in x^μ identity Z obeys. If ω^a is constant this reduces to the global G WI. Let's check the structure of this identity more closely.

First let's recall that $\mathcal{L}_{inv}(\varphi, A_\mu)$ is gauge invariant thus

$$\delta_Q(\omega)\mathcal{L} = \delta_Q(\omega)\mathcal{L}_f + \delta_Q(\omega)\mathcal{L}_{\phi\pi}$$

Now

$$\delta_Q(\omega)\mathcal{L}_f = -f_a\,\delta_Q(\omega) f_a$$

$$\delta_Q(\omega)\mathcal{L}_{\phi\pi} = \int d^4y\,\bar{c}_a(x)\left(\delta_Q(\omega) M_f^{ab}(x,y)\right)c_b(y)$$

recall that

$$\frac{\delta_Q(\omega) f_a(x)}{\delta\omega^b(y)} = M_f^{ab}(x,y)$$

thus

$$\delta_Q(\omega) f_a(x) = \int d^4y\, M_f^{ab}(x,y)\,\omega^b(y)$$

for infinitesimal $\omega^a(x)$.

So we find

$$\delta_Q(\omega)\mathcal{L}_f = -\int d^4y\, f_a(x)\, M_f^{ab}(x,y)\,\omega^b(y)$$

$$\delta_Q(\omega)\mathcal{L}_{\phi\pi} = \int d^4y\,\bar{c}_a(x)\left(\delta_Q(\omega) M_f^{ab}(x,y)\right)c_b(y)$$

Thus we find the Gauge WI

$$\boxed{\begin{aligned}&\int d^4x\left[J_\alpha\,\delta_Q(\omega)\varphi^\alpha + J^a_\mu\,\delta_Q(\omega)A^{a\mu}\right]Z[J,J_\mu,\xi,\bar\xi]\\ &= \int d^4x\,d^4y\Big[f_a(x)M_f^{ab}(x,y)\,\omega^b(y)\\ &\qquad - \bar c_a(x)\left(\delta_Q(\omega)M_f^{ab}(x,y)\right)c_b(y)\Big]\times\\ &\qquad\qquad \times Z[J,J_\mu,\xi,\bar\xi]\end{aligned}}$$

If a theory is to be gauge invariant the generating functional must obey this gauge WI and then the S-matrix will exhibit this local gauge invariance.

Of course this is messy compared to a global WI since the RHS is not zero for a gauge invariant theory i.e. $\delta_Q(\omega)\mathcal{L} \neq 0$,