for the Coulomb gauge

$$f_a(A_\mu) = \partial_i A_i^a$$

and

$$\delta_Q(\omega) f_a(A_\mu)(x) = \partial_i^x\left(\partial_i \omega^a(x) - f_{abc} A_i^b(x)\omega^c(x)\right)$$

since $\partial_i A_i^a = 0 \Rightarrow$ $= \nabla_x^2 \omega^a(x) - f_{abc} A_i^b(x) \partial_i^x \omega^c(x)$

So

$$M_f^{ac}(x,y) = \frac{\delta_Q(\omega) f_a(A_\mu)(x)}{\delta \omega^c(y)}$$

$$= \left[\nabla_x^2 \delta^{ac} - f_{abc} A_i^b(x) \partial_i^x\right] \delta^4(x-y)$$

$$= M_C^{ac}(x,y)$$

This is precisely the matrix factor we derived rigorously in the Coulomb gauge treatment from first principles. Hence the intuitive approach checks with the rigorous derivation.

Finally let's consider the equivalence of the Coulomb gauge integral to the integral in a manifestly Lorentz invariant gauge

The Coulomb gauge generating functional is given by

$$Z_C[j_\mu^a] = \int [dA_\mu^a]\, \Delta_C[A_\mu]\, \delta[\partial_i A_i^a]\, e^{i\int d^4x [\mathcal{L}_{inv}(A) + j_\mu^a A^{a\mu}]}$$

"C" for Coulomb gauge

with $j_0^a = 0$ understood and $\partial_i j_i^a = 0$; we only couple to the physical degrees of freedom. (Of course we can also consider Green functions involving all components of A_μ^a; however the physical states & hence S-matrix involve only the transverse degrees of freedom)

We again insert $1 = \Delta_f[A_\mu] \int [dg]\, \delta[f_a(A_\mu^g)]$

into the integral

$$Z_C[j_\mu^a] = \int [dA_\mu^a] \int [dg]\, \delta[f_a(A_\mu^g)]\, \Delta_f[A_\mu]\, \Delta_C[A_\mu]\, \delta[\partial_i A_i^a]\, e^{i\int d^4x [\mathcal{L}_{inv}(A) + j_\mu^a A^{a\mu}]}$$

As usual we make a change of variables corresponding to a gauge transformation on $A_\mu^a \to A_\mu^{ag}$

$$Z_c[j^a_\mu] = \int [dA^{ag'}_\mu] \int [dg]\, \delta[f_a(A^{g'g}_\mu)]\, \Delta_f[A^{g'}_\mu]\, \Delta_c[A^{g'}_\mu]$$
$$\delta[\partial_i A^{ag'}_i]\; e^{i\int d^4x [\mathcal{L}_{inv}(A^{g'}) + j^a_\mu A^{a\mu g'}]}$$

But recall that $\mathcal{L}_{inv}(A^{g'}) = \mathcal{L}_{inv}(A)$ is gauge invariant as well as the measure $[dA^{ag'}_\mu] = [dA^a_\mu]$ and further

$$\Delta_f[A^{g'}_\mu] = \Delta_f[A_\mu]$$

$$\Delta_c[A^{g'}_\mu] = \Delta_c[A_\mu]$$

So for $g' = g^{-1}$ then $\delta[f_a(A^{g'g}_\mu)] = \delta[f_a(A_\mu)]$

we have

$$Z_c[j^a_\mu] = \int [dA^a_\mu]\, \delta[f_a(A_\mu)]\, \Delta_f[A_\mu] \times$$
$$\times\, e^{i\int d^4x [\mathcal{L}_{inv}(A) + j^a_\mu A^{a\mu}]} \times$$
$$\times \int [dg]\, \delta[\partial_i A^{ag^{-1}}_i]\, \Delta_c[A_\mu]\, e^{i\int d^4x\, [j^a_\mu (A^{a\mu g^{-1}} - A^{a\mu})]}$$

Now since the g-integral is multiplied by $\delta[f_a(A_\mu)]$ we only need to evaluate it for gauge fields obeying the constraint $f_a(A_\mu)=0$. Thus according to the $\delta[\partial_i A_i^{ag^{-1}}]$ we must evaluate the g-integral at some g_0 such that, for $f_a(A_\mu)=0$, we have:

$$0=\partial_i A_i^{g_0} = \partial_i\left[U(g_0)\,A_i\,U^{-1}(g_0) + U(g_0)\partial_i U^{-1}(g_0)\right]$$

we can solve this equation for $U(g_0)$ as a power series in A_μ. In lowest order we find that

$$A_i^{g_0} = \left(\delta_{ij} - \frac{\partial_i\partial_j}{\partial^2}\right)A_j + O(A_\mu^2)$$

(Exercise – find the first couple of terms in this expansion).

So the g-integral can be evaluated

using the expansion of $\partial_i A_i^{ag^{-1}}$ about g_0

$$\partial_i A_i^{ag^{-1}} = \partial_i A_i^{ag_0} - \delta_Q(\omega)\partial_i A_i^{ag_0}$$

where $\partial_i A_i^{ag_0} = 0$

hence

$$\int [dg]\,\delta[\partial_i A_i^{ag^{-1}}]\, e^{i\int d^4x\,[j_\mu^a (A^{a\mu g^{-1}} - A^{a\mu})]}$$

$$= \int [d\omega^b]\,\delta[-\delta_Q(\omega)\partial_i A_i^{ag_0}]\, e^{i\int d^4x\,[j_\mu^a (A^{a\mu g_0} - A^{a\mu})]}$$

$$= \frac{1}{\det\left|\dfrac{\delta_Q(\omega)\partial_i A_i^{ag_0}(x)}{\delta\omega^b(y)}\right|}\, e^{i\int d^4x\,[j_\mu^a (A^{a\mu g_0} - A^{a\mu})]}$$

but

$$\Delta_c[A_\mu^{g_0}] = \det\left|\frac{\delta_Q(\omega)\partial_i A_i^{ag_0}(x)}{\delta\omega^b(y)}\right|$$

$$= \Delta_c[A_\mu] \text{ since}$$

Δ_c is gauge invariant.

So we find

$$Z_C[j^a_\mu] = \int [dA^a_\mu]\, \delta[f_a(A_\mu)]\, \Delta_f[A_\mu] \times$$

$$\times\, e^{i\int d^4x\, [\mathcal{L}_{inv}(A) + j^a_\mu A^{a\mu}]} \times$$

$$\times\, \cancel{\Delta_C[A_\mu]\, \Delta^{-1}_C[A^{go}_\mu]}\, e^{i\int d^4x\, j^a_\mu (A^{a\mu go} - A^{a\mu})}$$

$$\boxed{Z_C[j^a_\mu] = \int [dA^a_\mu]\, \delta[f_a(A_\mu)]\, \Delta_f[A_\mu]\, e^{i\int d^4x (\mathcal{L}_{inv}(A) + j^a_\mu A^{a\mu})} \times\, e^{i\int d^4x (A^{a\mu go} - A^{a\mu})}}$$

Now A^{go}_μ is a power series in A_μ as we stated with

$$A^{go}_i = \left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) A_j + O(A^2_\mu)$$

Further $j^a_0 = 0$ and $\partial_i j^a_i = 0$ for the Coulomb gauge $\Downarrow$

$$\left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) j^a_j = j^a_i$$

So

$$j^a_\mu A^{a\mu g_0} = j^a_i A^{a i g_0}$$

$$= j^a_i\left[\left(\delta_{ij} - \frac{\partial_i \partial_j}{\nabla^2}\right) A^a_j + O(A_\mu^2)\right]$$

So integrating by parts we have

$$\int d^4x\, j^a_\mu A^{a\mu g_0} = \int d^4x\, j^a_i \left[A^a_i + O(A_\mu^2)\right]$$

$$= \int d^4x\, j^a_\mu \left[A^{a\mu} + O(A_\mu^2)\right]$$

So defining this as $\equiv \int d^4x\, j^a_\mu \left[A^{a\mu} + F^{a\mu}(A)\right]$

we obtain

$$Z_C[j^a_\mu] = \int [dA^a_\mu]\, \delta[f_a(A_\mu)]\, \Delta_f[A_\mu] \times$$
$$\times e^{i\int d^4x\, j^a_\mu F^{a\mu}(A)}\; e^{i\int d^4x\,(\mathcal{L}_{inv}(A) + j^a_\mu A^{a\mu})}$$

$$\boxed{Z_C[j^a_\mu] = e^{i\int d^4x\, j^a_\mu F^{a\mu}\left[\frac{\delta}{i\delta j^a_\mu}\right]}\; Z_f[j^a_\mu]}$$

where

$$\boxed{\begin{aligned} Z_f[j^a_\mu] = \int [dA^a_\mu]\, \delta[f_a(A_\mu)]\, \Delta_f[A_\mu] \times \\ \times\, e^{i\int d^4x \left[\mathcal{L}_{inv}(A) + j^a_\mu A^{a\mu}\right]} \end{aligned}}$$

is the $f_a(A^\mu)=0$ gauge generating functional.

So the Coulomb generating functional is related to the arbitrary $f^a=0$ gauge functional by a field redefinition of A^a_μ

i.e.

$$\langle 0|T\, A^a_i \cdots A^b_j|0\rangle_{\text{coulomb}}$$

$$= \langle 0|T\left(A^a_i + O(A^2)\right)\cdots\left(A^b_j + O(A^2)\right)|0\rangle_f$$

Hence we see immediately that the S-matrices calculated in either gauge are the same since $\langle 0|T\cdots|0\rangle_{\text{coulomb}}$ has the same poles as $\langle 0|T\cdots|0\rangle_f$ the only difference being the Residue at the pole - but the S-matrix reduction formula divides out the residue factors due to the asymptotic condition. Thus

$$S_{\text{coulomb}} = S_{f\text{-gauge}} .$$

The long and short of this is we can calculate $Z[j^a_\mu]$ in any gauge we desire and with any arbitrary source j^a_μ (not necessarily $j^a_0 = 0$, $\partial_i j^a_i = 0$), then in order to extract the S-matrix