Before considering whether we can extend a global invariance to a local invariance let's consider an explicit example of a theory with a global internal symmetry (as well as global Poincaré invariance)

Example: $SU(2)_L \times SU(2)_R$ σ-model.

We would like to consider a model in which the fields belong to representations of $SU(2)_L \times SU(2)_R$. Since the Lie algebra for $SU(2)_L \times SU(2)_R$ is the same as that for $O(4)$ the simplest set of fields we can imagine that transform non-trivially under $O(4)$ is simply the fundamental or $O(4)$ vector representation. This consists of 4-Hermitian scalar fields ϕ^i ; $i=1,2,3,4$ that rotate into each other under 4-dimensional "isospin" transformations

$$\mathcal{U}(\omega) = e^{-i\omega^a Q^a} \equiv e^{-i\frac{\omega^{ij}}{2} Q^{ij}} \quad a = 1,2,3,4,5,6$$

6 independent ∡'s of rotation in 4 dimensional Euclidean space, or $i,j = 1 \ldots 4$

$\omega^{ij} = -\omega^{ji}$

where the commutation relations of $O(4)$ can be found by recalling that if α^i are the coordinates of 4-dimensional Euclidean space then under 4D dimensional rotations they transform as (just like relativity)

$$\equiv \alpha^k - \omega^{km}\alpha^m$$

$$\alpha'^k = \alpha^k + \frac{\omega^{ij}}{2}\left(\alpha^i \partial_j - \alpha^j \partial_i\right)\alpha^k$$

Thus

$$= \alpha^k + \frac{\omega^{ij}}{2}\left(\alpha^i \delta^{jk} - \alpha^j \delta^{ik}\right)$$

$$= \alpha^k + \frac{\omega^{ij}}{2}\left(\delta^{im}\delta^{jk} - \delta^{ik}\delta^{jm}\right)\alpha^m = \alpha^k + \omega^{mk}\alpha^m$$

$$\equiv \alpha^k - i\frac{\omega^{ij}}{2}\left(T^{ij}\right)_{km}\alpha^m \; ; \quad \text{i.e. } \left(T^{ij}\right)_{km} \equiv i\left(\delta^{im}\delta^{jk} - \delta^{ik}\delta^{jm}\right)$$

So

$$\left[T^{ij}, T^{kl}\right]_{mn} = \left(T^{ij}\right)_{mp}\left(T^{kl}\right)_{pn} - \left(T^{kl}\right)_{mp}\left(T^{ij}\right)_{pn}$$

$$= i^2\left(\delta^{ip}\delta^{jm} - \delta^{im}\delta^{jp}\right)\left(\delta^{kn}\delta^{lp} - \delta^{kp}\delta^{ln}\right) - \rightleftarrows$$

$$= \left(\delta^{il}\delta^{jm}\delta^{kn} - \delta^{ik}\delta^{jm}\delta^{ln} - \delta^{im}\delta^{jl}\delta^{kn} + \delta^{im}\delta^{jk}\delta^{ln} - \delta^{kj}\delta^{lm}\delta^{in} + \delta^{ki}\delta^{lm}\delta^{jn} + \delta^{km}\delta^{lj}\delta^{in} - \delta^{km}\delta^{li}\delta^{jn}\right) i^2$$

$$= \Big[\delta^{ik}\left(\delta^{jn}\delta^{lm} - \delta^{jm}\delta^{ln}\right) + \delta^{jl}\left(\delta^{in}\delta^{km} - \delta^{im}\delta^{kn}\right) - \delta^{il}\left(\delta^{jn}\delta^{km} - \delta^{jm}\delta^{kn}\right) - \delta^{jk}\left(\delta^{in}\delta^{lm} - \delta^{im}\delta^{ln}\right)\Big] i^2$$

$$[T^{ij}, T^{kl}]_{mn} = i\left[\delta^{ik}(T^{jl})_{mn} + \delta^{jl}(T^{ik})_{mn} - \delta^{il}(T^{jk})_{mn} - \delta^{jk}(T^{il})_{mn}\right]$$

So we have for the charges $Q^{ij} \sim (T^{ij})$

$$[Q^{ij}, Q^{kl}] = i\left[\delta^{ik} Q^{jl} + \delta^{jl} Q^{ik} - \delta^{il} Q^{jk} - \delta^{jk} Q^{il}\right].$$

And the fields transform like the coordinates did:

$$[Q^{ij}, \phi^{l}] = -(T^{ij})_{lm}\phi^{m} \equiv -i\delta_{Q}^{ij}\phi^{l}$$

Or for finite rotations

$$\mathcal{U}^{-1}(\omega)\,\phi^{i}(x)\,\mathcal{U}(\omega) = U(\omega)_{ij}\,\phi^{j}(x)$$

$$U_{ij}(\omega) = \left(e^{-i\frac{\omega^{kl}}{2}T^{kl}}\right)_{ij}$$

If this $SU(2)\times SU(2)$ symmetry is a good symmetry then

$$S = \mathcal{U}^{-1}(\omega)\, S\, \mathcal{U}(\omega) \ .$$ Since the

asymptotic states transform like 4 dimensional Euclidean basis vectors the ~~transition~~ amplitudes for various processes will be related. For example consider $|\vec{p}_1, i_1; \vec{p}_2, i_2 \text{ in}\rangle$ to go into $|\vec{q}_1, j_1; \vec{q}_2\, j_2 \text{ out}\rangle$

where $$|\vec{p}, i \ {}^{\text{in}}_{\text{out}}\rangle = a^{\dagger i}_{\substack{\text{in}\\ \text{out}}}(\vec{p})\,|0\rangle \text{ etc.}$$

Then

$$S(i_1, i_2; j_1, j_2) = \langle \vec{q}_1, j_1; \vec{q}_2, j_2 \text{ out} | \vec{p}_1, i_1; \vec{p}_2, i_2 \text{ in}\rangle$$

$$= \langle \vec{q}_1, j_1; \vec{q}_2, j_2 \text{ in} | S | \vec{p}_1, i_1; \vec{p}_2, i_2 \text{ in}\rangle$$

$$= \langle \vec{q}_1, j_1; \vec{q}_2, j_2 \text{ in} | \mathcal{U}^{-1}(\omega)\, S\, \mathcal{U}(\omega) | \vec{p}_1, i_1; \vec{p}_2, i_2 \text{ in}\rangle$$

but $$\mathcal{U}(\omega)|\vec{p}_1, i_1, \vec{p}_2, i_2 \text{ in}\rangle = U(\omega)_{i_1' i_1}\, U(\omega)_{i_2' i_2}\, |\vec{p}_1, i_1'; \vec{p}_2, i_2' \text{ in}\rangle$$

So

$$\boxed{S(i_1,i_2;j_1,j_2) = U(\omega)_{i_1' i_1} U(\omega)_{i_2' i_2} U^{-1}(\omega)_{j_1 j_1'} U^{-1}(\omega)_{j_2 j_2'} S(i_1',i_2';j_1',j_2')}$$

Now for infinitesimal $\omega \Rightarrow$

$$S(i_1,i_2;j_1,j_2) = S(i_1,i_2;j_1,j_2) + \omega^{i_1' i_1} S(i_1',i_2;j_1,j_2) + \omega^{i_2' i_2} S(i_1,i_2';j_1,j_2) - \omega^{j_1 j_1'} S(i_1,i_2;j_1',j_2) - \omega^{j_2 j_2'} S(i_1,i_2;j_1,j_2')$$

Now $\Rightarrow$ a relation among S-matrix els.

$$0 = \omega^{i i_1} S(i,i_2;j_1,j_2) + \omega^{i i_2} S(i_1,i;j_1,j_2) + \omega^{i j_1} S(i_1,i_2;i,j_2) + \omega^{i j_2} S(i_1,i_2;j_1,i)$$

This can be checked experimentally.

for example
Suppose we consider $(\vec{p}_1,4)(\vec{p}_2,4) \to (\vec{q}_1,4)(\vec{q}_2,1)$

$(\vec{q}_1,4)(\vec{q}_2,2)$

and $\omega^{13} = -\omega^{23} \neq 0$ only as a specific example of $j_2 = 3$

$\Rightarrow$

$$0 = S(4,4;4,1)\,\omega^{13} + S(4,4;4,2)\,\omega^{23}$$

$$\Rightarrow \quad S(4,4;4,2) = S(4,4;4,1)$$

$$\Rightarrow \quad d\sigma((4,4)\to(1,4)) = d\sigma((4,4)\to(2,4)).$$

So from an experimental point of view we may discover a symmetry by noting various relations among cross sections.

As we know from the QAP, in order to build a theory with this invariance the O(4) WI's must be true and hence the Lagrangian describing the dynamics of the fields ϕ^i must be O(4) invariant. The $(\text{distance})^2$ in 4-dimensional Euclidean space is invariant under rotations — hence the Lagrangian is given in terms of this distance with restriction that we have at most dimension 4 in the fields to keep the theory renormalizable. Thus the most general O(4) invariant, renormalizable Lagrangian is given by

$$\mathcal{L} = \frac{Z}{2}\partial_\mu\phi^i\partial^\mu\phi^i - \frac{(m^2+a)}{2}\phi^i\phi^i - \frac{(\lambda+c)}{8}(\phi^i\phi^i)^2$$

where again $a, b = Z-1, c$ are power series in λ to be specified. We can check the $\mathcal{L}$ is invariant

$$\frac{\omega^{ij}}{2}\Big[\delta_Q^{ij}\mathcal{L} = Z\,\partial_\mu\delta_Q^{ij}\phi^k\partial^\mu\phi^k - (m^2+a)\phi^k\delta_Q^{ij}\phi^k$$

$$- \frac{(\lambda+c)}{2}(\phi^l\phi^l)\,\phi^k\delta_Q^{ij}\phi^k\Big]$$

but $\frac{\omega^{ij}}{2}\delta_Q^{ij}\phi^k = \omega^{mk}\phi^m$

& $\frac{\omega^{ij}}{2}\delta_Q^{ij}\partial^\mu\phi^k = \omega^{mk}\partial^\mu\phi^m$ since this is a global symmetry hence $\partial^\mu\omega^{ij} \equiv 0$.

So $$\frac{\omega^{ij}}{2}\delta_Q^{ij}\left(\frac{\phi^k\phi^k}{2}\right) = \phi^m\omega^{mk}\phi^k = 0$$

$$\frac{\omega^{ij}}{2}\delta_Q^{ij}\,\frac{1}{2}\partial_\mu\phi^k\partial^\mu\phi^k = \partial_\mu\phi^m\omega^{mk}\partial^\mu\phi^k = 0$$

Since $\omega^{ij} = -\omega^{ji}$ and $\phi^m \phi^h$ is symmetric as is $\partial_\mu \phi^m \partial^\mu \phi^h$, so as stated

$$\delta_Q^{ij} \mathcal{L} = 0 \quad , \; \mathcal{L} \text{ is } O(4) \text{ invariant}.$$

The Green function generating functional is defined by

$$Z[J^i] = \int [d\phi^i]\, e^{i\int dx (\mathcal{L}_\phi + J^i \phi^i)} .$$

The Schwinger action principle implies that O(4) WI is valid

$$\delta_Q^{ij} Z[J] = 0$$

where
$$\delta_Q^{ij} \equiv \int dx\, J^k(x)\, \delta_Q^{ij} \frac{\delta}{\delta J^h(x)}$$
$$= \int dx\, J^k(x) (-iT^{ij})_{k\ell} \frac{\delta}{\delta J^\ell(x)}$$
$$= \int dx \left[J^j \frac{\delta}{\delta J^i} - J^i \frac{\delta}{\delta J^j} \right] ,$$

Hence the Green functions obey

$$0 = \sum_{i=1}^{n} \langle 0 | T \phi^{i_1}(x_1) \cdots [\phi^i \delta^{j i_i} - \phi^j \delta^{i i_i}](x_i) \cdots \phi^{i_n}(x_n) | 0 \rangle$$

We could calculate the Noether current and obtain the local O(4) variation of the generating functional

$$\partial_\mu J^\mu_{ij}\left[\frac{\delta}{i\delta J^i}\right](x)\, Z[J]$$
$$= -i\,\delta_Q^{ij}(x)\, Z[J].$$

where recall

$$J^\mu_{ij} \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi^k}\, \delta_Q^{ij}\, \varphi^k$$

$$= Z\partial^\mu \varphi^k \delta_Q^{ij} \varphi^k$$

$$= Z\partial^\mu \varphi^k\left[-i(T^{ij})_{kl}\varphi^l\right]$$

$$= Z\,\partial^\mu \varphi^k\left(\delta^{il}\delta^{jk} - \delta^{ik}\delta^{jl}\right)\varphi^l$$

$$= Z\left(\partial^\mu \varphi^j \varphi^i - \partial^\mu \varphi^i \varphi^j\right)$$

$$\boxed{J^\mu_{ij} = Z\,\varphi^i \overleftrightarrow{\partial^\mu} \varphi^j}$$

Of course we have formally derived these relations exploiting the properties of the path integral representation of $Z[J]$. In addition we could check explicitly that these $O(4)$ current conservation and WI relations are valid within perturbation theory. Suppose we treat λ as the perturbation parameter so that

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$$

with $\mathcal{L}_I \equiv \frac{b}{2}\partial_\mu \varphi^i \partial^\mu \varphi^i - \frac{a}{2}\varphi^i\varphi^i - \frac{\lambda+c}{8}(\varphi^i\varphi^i)^2$

Then we have the Gell-Mann-Low expansion for the Green functions

$$Z[J] = e^{i\int dx\, \mathcal{L}_I\left[\frac{\delta}{i\delta J^i}\right]} Z_{in}[J]$$

with

$$Z_{in}[J] = e^{-\frac{1}{2}\int dx dy\, J^i(x)\Delta_F^{ij}(x-y) J^j(y)}$$

and $$\Delta_F^{ij}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i\delta^{ij}}{p^2-m^2+i\epsilon}$$

determined from $\mathcal{L}_0$.

Re-expressing this as a sum over Feynman diagrams we find

$$\langle 0|T\,\phi^{i_1}(x_1)\cdots\phi^{i_n}(x_n)|0\rangle$$

$$= \int\frac{d^4p_1}{(2\pi)^4}\cdots\frac{d^4p_n}{(2\pi)^4}\, e^{-i\sum_i p_i x_i}\sum_{\Gamma\in G^n}(2\pi)^4\delta^4(p_{11}+\cdots)\cdots$$

$$\cdots(2\pi)^4\delta^4(p_{a1}+\cdots)$$

$$\times\,\alpha(\Gamma)\int\frac{d^4k_1}{(2\pi)^4}\cdots\frac{d^4k_{m(\Gamma)}}{(2\pi)^4}\,I_\Gamma(p,k)$$

where 1) G^n is the set of all topologically distinct diagrams with n-external ϕ-lines made from the lines and vertices listed below and with the external and internal momentum routed through the diagram with 4-m conservation at each vertex.

2) Γ consists of "a" connected subdiagrams.

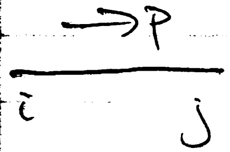
3) $\alpha(\Gamma)$ is the combinatoric factor needed to go from Wick's theorem to a sum over topologically distinct diagrams

4) $m(\Gamma)$ = # of independent loops belonging

to diagram Γ

5) $I_\Gamma(p,k)$ is the Feynman integrand made from the product of lines and vertex factors as listed below

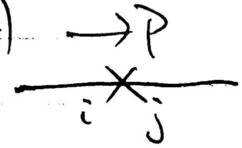
1) Lines — Propagator Factor



$$\frac{i\delta^{ij}}{p^2-m^2+i\epsilon}$$

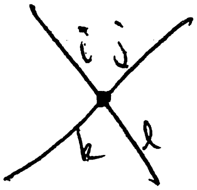
2) Vertices — Vertex Factor

a)



$$+ib\delta^{ij}p^2-ia\delta^{ij}$$

b)

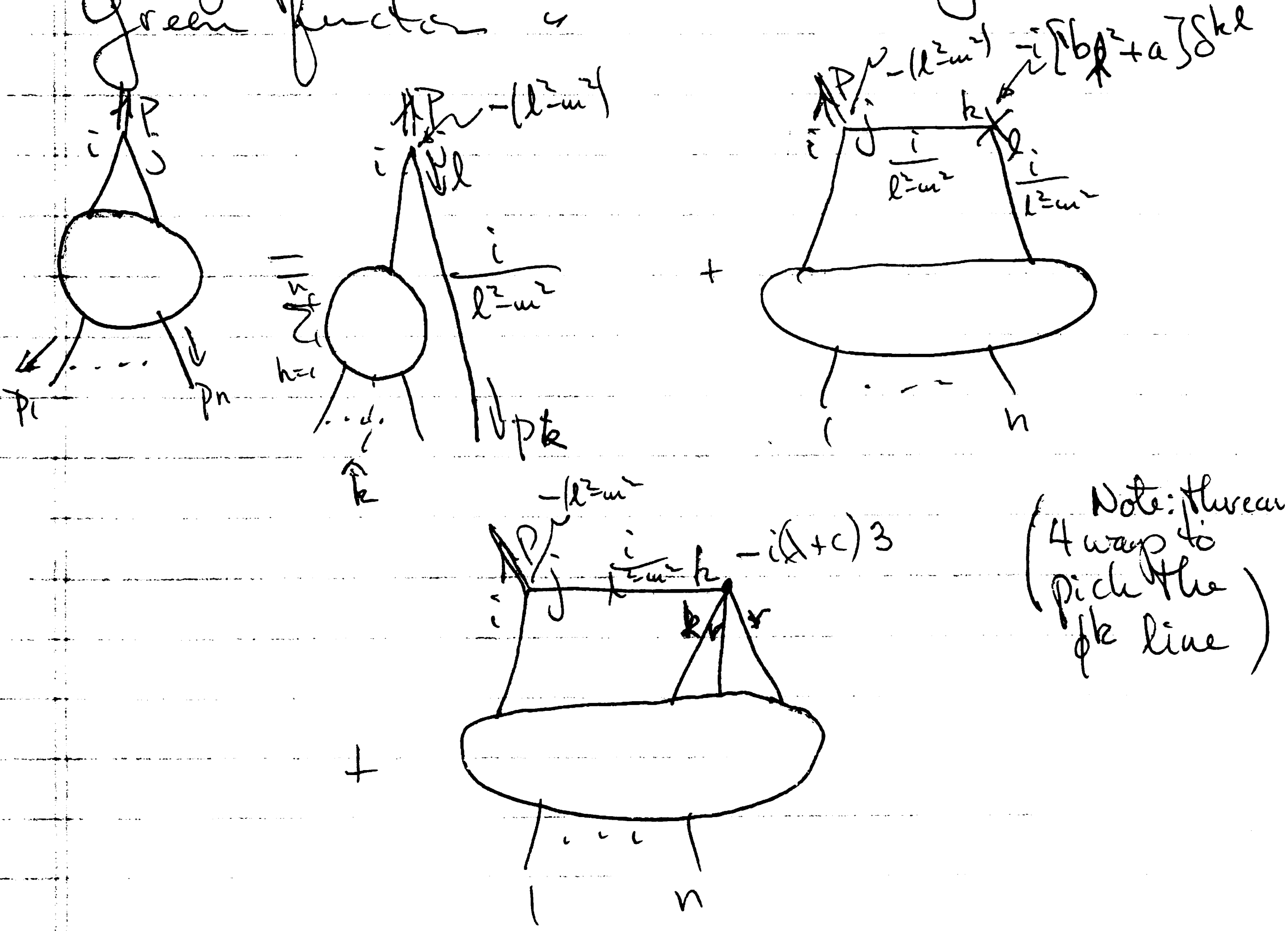


$$-i(\lambda+c)[\delta^{ij}\delta^{kl}+\delta^{ik}\delta^{jl}+\delta^{il}\delta^{jk}]$$

We can then verify explicitly that the field equations are satisfied — in particular the bi-linear field equations

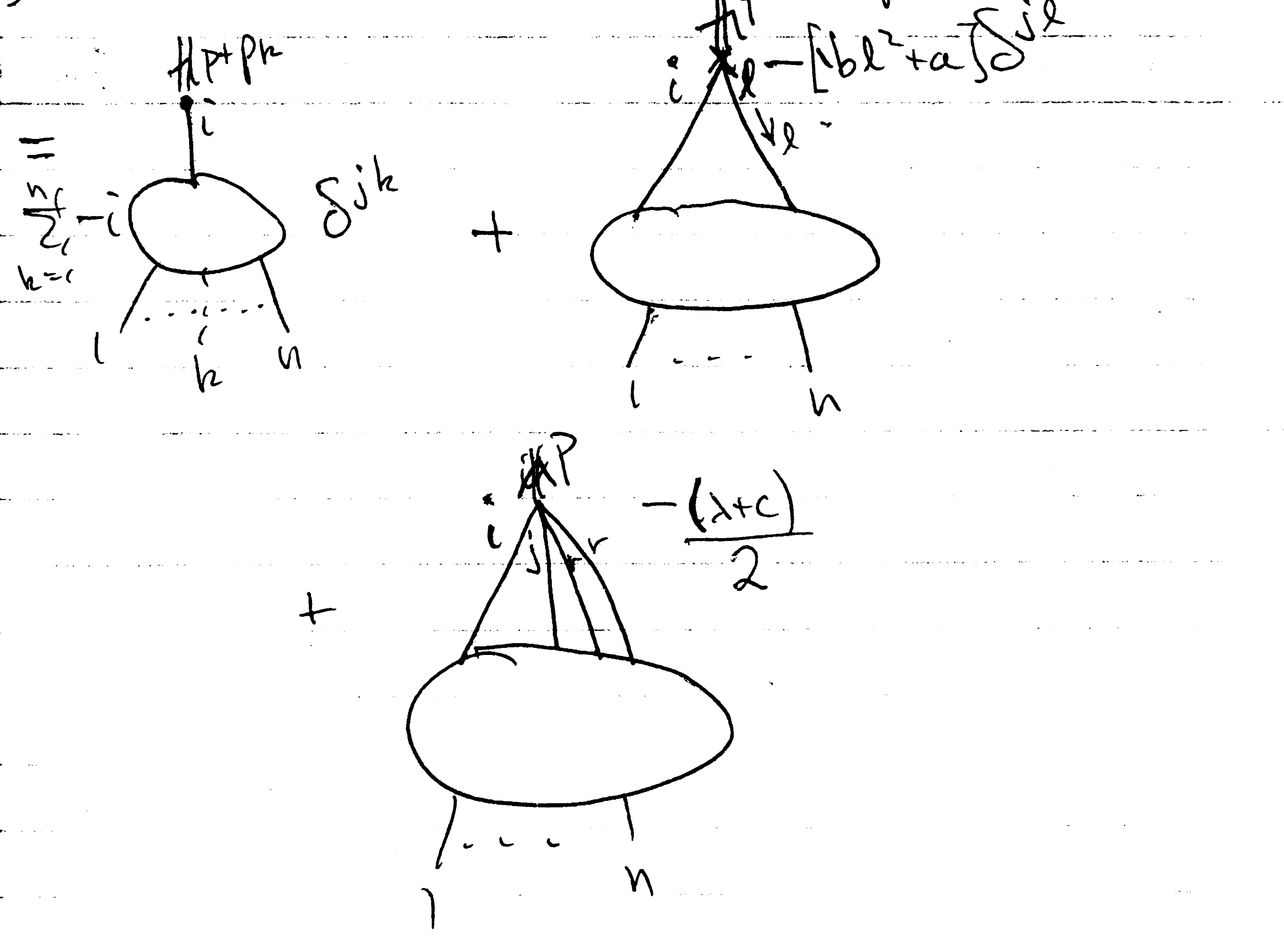
$$[\phi^i(\partial^2+m^2)\phi^j](x)\, Z[J]$$

The graphical structure contributing to this Green function is



(Note: there are 4 ways to pick the ϕk line)

The $-(l^2-m^2)$ factor at the vertex cancels the propagator $\frac{i}{l^2-m^2}$ leaving a $(-i)$ factor and a graphical structure that looks like the $\frac{i}{l^2-m^2}$ line contracted to a point



In coordinate space these are graphical contributions to the Green functions

$$\langle 0|T\left[\phi^i(\partial^2+m^2)\phi^j\right](x)\, \underbrace{\phi^{i_1}(x_1)\cdots\phi^{i_n}(x_n)}_{\equiv X}|0\rangle$$

$$= \sum_{i=1}^{n} -i\delta^4(x-x_i)\delta^{j i_i}\langle 0|T\phi^{i_1}(x_1)\cdots\hat{\phi}^{i}(x_i)\cdots\phi^{i_n}(x_n)|0\rangle$$

$$-a\langle 0|T[\phi^i\phi^j](x)\, X|0\rangle - b\langle 0|T[\phi^i\partial^2\phi^j](x)\, X|0\rangle$$

$$-\frac{(\lambda+c)}{2}\langle 0|T[\phi^i\phi^j\phi^m\phi^m](x)\, X|0\rangle$$

$\Rightarrow$ The bi-linear field equation:

$$\boxed{\begin{aligned}&\langle 0|T\left[\phi^i(Z\partial^2+m^2+a)\phi^j\right](x)\, X|0\rangle \\ &\quad+\frac{(\lambda+c)}{2}\langle 0|T[\phi^i\phi^j\phi^m\phi^m](x)\, X|0\rangle \\ &\qquad= -i\sum_{i=1}^{n}\delta^4(x-x_i)\delta^{j i_i}\langle 0|T\, X_{\hat{i_i}}|0\rangle\end{aligned}}$$

Differentiating by parts we have

$$\langle 0|T\,\partial_\mu[Z\phi^i\gamma^\mu\phi^j](x)\,X|0\rangle$$
$$+\langle 0|T[-Z\partial_\mu\phi^i\gamma^\mu\phi^j+(m^2+a)\phi^i\phi^j](x)\,X|0\rangle$$
$$+\frac{(\lambda+c)}{2}\langle 0|T[\phi^i\phi^j\phi^m\phi^m](x)\,X|0\rangle$$
$$=-i\sum_{i=1}^{n}\delta^4(x-x_i)\,\delta^{j i_i}\langle 0|T\,X_{\hat{i}_i}^{i}|0\rangle$$

Now we can multiply this identity by

$$(-iT^{kl})_{ij}=\delta^k{}_j\delta^l{}_i-\delta^k{}_i\delta^l{}_j$$

Which yields

$$-\partial_\mu\langle 0|T[Z\phi^k\overleftrightarrow{\partial}^\mu\phi^l](x)\,X|0\rangle$$

$$-\langle 0|T\,\delta_Q^{ij}\mathcal{L}(x)\,X|0\rangle$$
$$=-i\sum_{i=1}^{n}\delta^4(x-x_i)(-iT^{kl})_{i i_i}\langle 0|T\phi^{i_1}(x_1)\cdots\phi^{i}(x_i)\cdots\phi^{i_n}(x_n)|0\rangle$$

Of course we found $\delta_Q^{ij}\mathcal{L}=0$ hence

$$\boxed{\partial^x_\mu \langle 0|T\, J^\mu_{kl}(x)\, \Phi|0\rangle = i\sum_{i=1}^{n}\delta^4(x-x_i)(-iT^{kl})_{i_i i}\langle 0|T\phi^{i_1}(x_1)\cdots\phi^{i}(x_i)\cdots\phi^{i_n}(x_n)|0\rangle}$$

Multiplying by sources we find

$$\partial^x_\mu J^\mu_{kl}\left[\frac{\delta}{i\delta J^i}\right](x)\, Z[J]$$

$$= i \quad iJ^j(x)(-iT^{kl})_{ij}\frac{\delta}{i\delta J^i(x)}Z[J]$$

$$= -i \quad J^i(x)(-iT^{kl})_{ij}\frac{\delta}{\delta J^j(x)}Z[J]$$

$$= -i\delta_Q^{kl}(x)\, Z[J]$$

which is just Noether's theorem. Integrating over x we find the $O(4)$ Ward Identity

$$\delta_Q^{kl}Z[J]=0.$$

Since the states of our theory as well as the Lagrangian respect the O(4) symmetry & that is $Z[J]$ obeys the O(4) WI this is known as the Wigner-Weyl realization of the symmetry. Indeed the ground state $|0\rangle$ ~~energy~~ is the lowest and it is the invariant state defined by

$$\langle 0|\phi^i|0\rangle = 0 \text{ and the}$$

Hamiltonian consists of all positive terms for $\lambda + c \geq 0$, $m^2 + a \geq 0$, $Z \geq 1$

Suppose we choose $\underbrace{m^2 + a}_{\equiv -\mu^2;\ \mu^2>0} < 0$ and consider the ground state defined such that $\langle 0|\phi^i|0\rangle \equiv v^i = \overset{\text{spacetime}}{\text{constant}}$

The ground state expectation value of the energy density is just given by the lower bound of the potential energy with $\phi^i = v^i$

i.e.

$$\langle 0|\phi^i\phi^i|0\rangle = \langle 0|\phi^i|0\rangle\langle 0|\phi^i|0\rangle + \sum_{n\geq 1}\langle 0|\phi^i|n\rangle\langle n|\phi^i$$

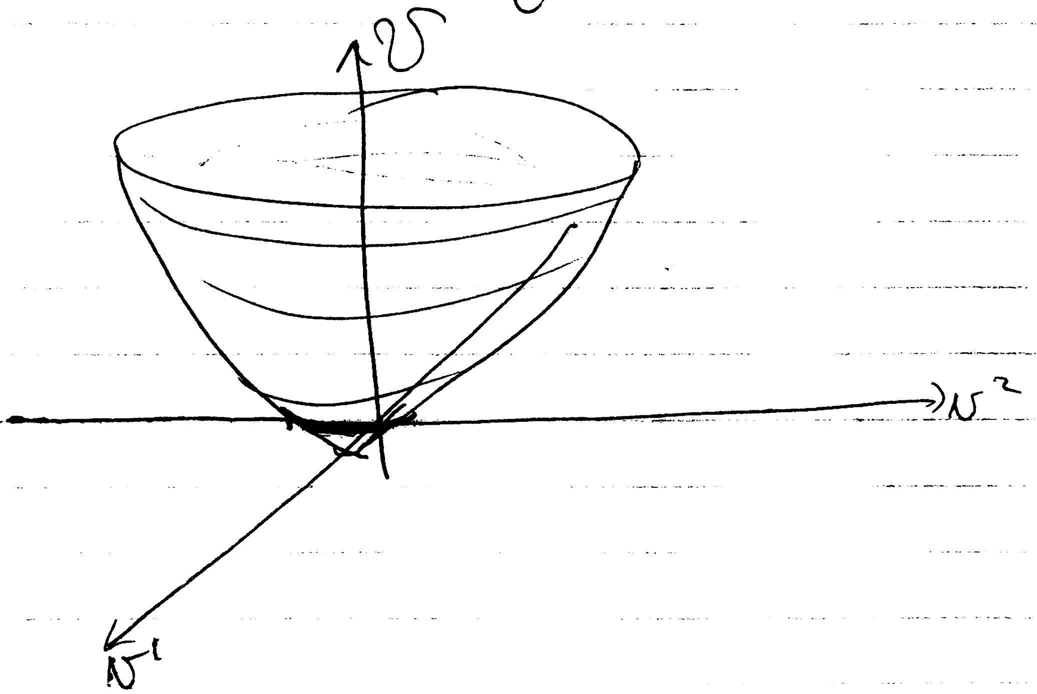
$$= v^i v^i + \sum_{n\geq 1}' |\langle 0|\phi^i|n\rangle|^2$$

$$\geq v^i v^i$$

Hence $\langle 0|\mathcal{H}|0\rangle \geq \mathcal{V}(v^i)$

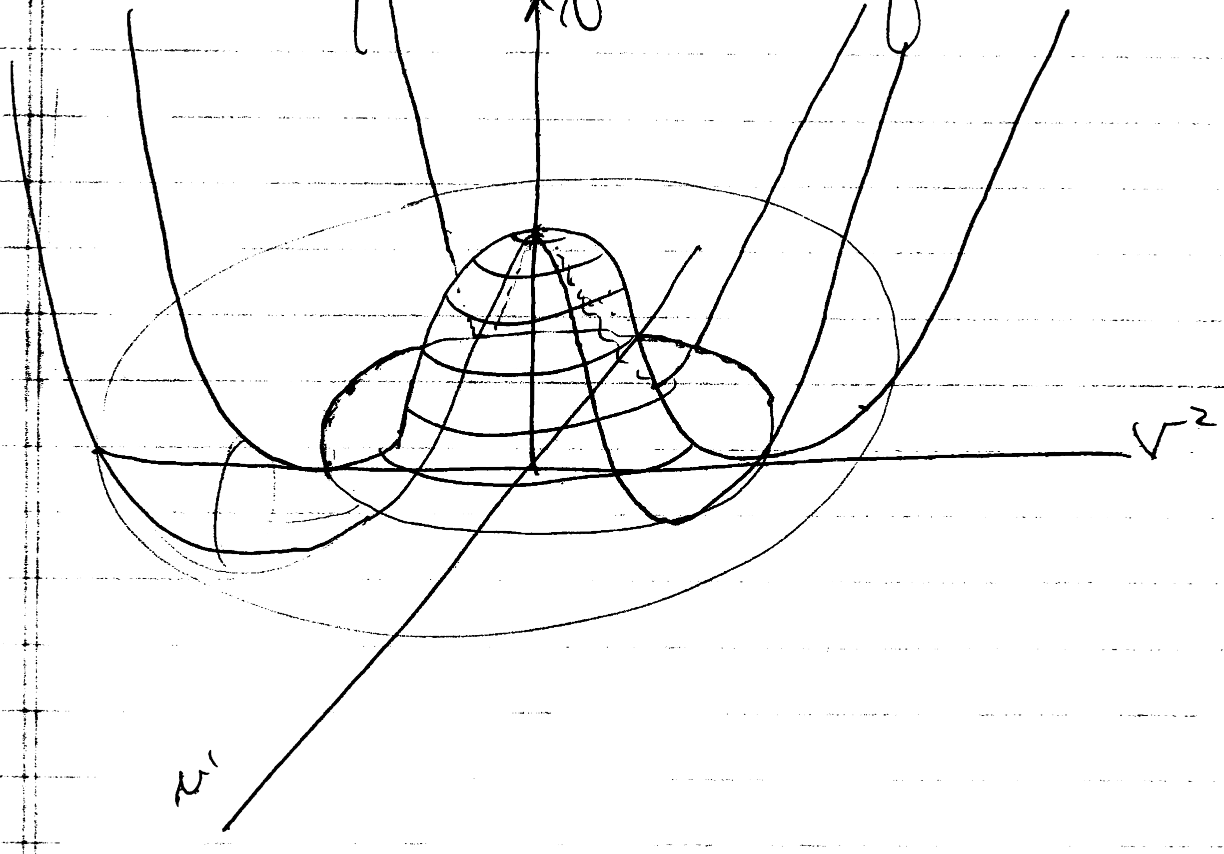
where $\mathcal{H} = \frac{1}{2z}\pi^i\pi^i + \mathcal{V}(\phi)$

Now we can plot $\mathcal{V}$ as a function of v^i for $m^2 + a \geq 0$ we find



The $v = 0$ value has the lowest bound on the energy, to which we usually normalize the energy - i.e. $\langle 0|\mathcal{H}|0\rangle = 0$.

However if $m^2+a<0$ we find



The potential has a relative maximum at $v=0$ and absolute minima at $v\neq 0$

Now let $a=c=0$ for convenience so

$$\mathcal{V} = +\frac{m^2}{2}\phi^i\phi^i + \frac{\lambda}{8}(\phi^i\phi^i)^2 \Big|_{\phi=v}$$

$$= \frac{v^iv^i}{2}\left[\frac{\lambda}{4}(v^jv^j)+m^2\right]$$

$$\frac{\partial V}{\partial v^l} = v^l\left[\frac{\lambda}{4}v^2 + m^2\right] + \frac{v^2}{2}\left[\frac{\lambda}{2}v^l\right]$$

$$= v^l\left[\frac{\lambda}{2}v^2 + m^2\right]$$

Thus $\frac{\partial V}{\partial v^l} = 0$ at $\boxed{\begin{array}{l} v^l = 0 \\ v^2 = \frac{-2m^2}{\lambda} \end{array}} = \frac{2\mu^2}{\lambda}$

$$\frac{\partial^2 V}{\partial v^l \partial v^k} = \delta^{lk}\left[\frac{\lambda}{2}v^2 + m^2\right] + \lambda v^l v^k$$

$$\left.\frac{\partial^2 V}{\partial v^l \partial v^k}\right|_{v=0} = +m^2\delta^{lk} < 0$$ so $v = 0$ is a relative max.

$$\left.\frac{\partial^2 V}{\partial v^l \partial v^k}\right|_{v^2 = \frac{2\mu^2}{\lambda}} = \lambda v^l v^k > 0$$ so $v^2 = \frac{-2m^2}{\lambda}$ is a relative min.

Hence the vacuum energy at $v = 0$ is greater than at $v^2 = -2m^2/\lambda = \frac{2\mu^2}{\lambda}$

Hence $\langle 0'|\phi^i|0'\rangle = 0$ defines an unstable ground state & it will decay into the $\langle 0|\phi^i|0\rangle = v^i$ defined vacuum.

Hence we define the true ground state of the system to be

$$\langle 0|\phi^i|0\rangle = v^i \quad \text{with } v^2 = \frac{2\mu^2}{\lambda}.$$

Note: this $|0\rangle$ is degenerate — ∞ # of ground states since $v^2 = \frac{2\mu^2}{\lambda}$ is any circle only. Use $O(4)$ transformation to rotate all v into one ϕ^i say ϕ^4 and choose the positive v.

$$\langle 0|\phi^i|0\rangle = \delta^{i4}\, v$$

and

$$v = +\sqrt{2\mu^2/\lambda}\,.$$

Hence we can introduce the field

$$\phi^i = \phi'^i + v\,\delta^{i4}$$

and "quantize about v" that is ϕ'^i will have zero expectation value $\langle 0|\phi'^i|0\rangle = 0$ and it describes the quantum fluctuations about v.

Note a couple of points

i) $|0\rangle$ is no longer invariant even though $\mathcal{L}$ is

Assume $\mathcal{U}|0\rangle = |0\rangle$

Then $\langle 0|\phi^i|0\rangle = \langle 0|\mathcal{U}^{-1}\phi^i\mathcal{U}|0\rangle$

$$= \langle 0|\phi^i|0\rangle + \langle 0|\delta_a\phi^i|0\rangle$$

$\Rightarrow \langle 0|\delta_a\phi^i|0\rangle = 0$

but $\delta_a\phi^i = \omega^{ji}\phi^j$

So $\langle 0|\omega^{ji}\phi^i|0\rangle = 0$

but $\langle 0|\phi^i|0\rangle = v^i \neq 0$

So we have a contradiction!

hence $\mathcal{U}|0\rangle \neq |0\rangle$, the ground state is not invariant even though the dynamics is $O(4)$ invariant.

This is called spontaneous symmetry breakdown — or the Goldstone mode of Realizing the O(4) symmetry.

For $m^2 < 0$ this is the O(4) Goldstone Model.
$m^2 = -\mu^2$

2) The Lagrangian can be re-written in terms of the "shifted" field variable

$$\phi^i = \phi'^i + v^i$$

Since v^i is a constant $\partial_\mu \phi^i = \partial_\mu \phi'^i$

and

$$\phi^i\phi^i = \phi'^i\phi'^i + v^i v^i + 2v^i\phi'^i$$

$$(\phi^i\phi^i)^2 = (\phi'^i\phi'^i)^2 + (v^2)^2 + 4(v\cdot\phi')^2 + 2v^2\phi'^2 + 4v^2 v\cdot\phi' + 4\phi'^2 v\cdot\phi'$$

So

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi'^i\partial^\mu\phi'^i + \frac{1}{2}\mu^2\phi'^2 + \frac{v^2\mu^2}{2} + \mu^2 v\cdot\phi' - \frac{\lambda}{8}\Big[(\phi'^2)^2 + (v^2)^2 + 4(v\cdot\phi')^2 + 2v^2\phi'^2 + 4v^2 v\cdot\phi' + 4\phi'^2 v\cdot\phi'\Big]$$

Gathering like powers of ϕ' we have

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi'^i \partial^\mu \phi'^i + \overset{\to 0}{\left[\mu^2 - \frac{\lambda v^2}{2}\right]} v\cdot\phi'$$
$$+ \frac{1}{2}\left[\mu^2 - \frac{\lambda v^2}{2}\right]\phi'^2 - \frac{\lambda}{2}(v\cdot\phi')^2$$
$$- \frac{\lambda}{8}\left[(\phi'^2)^2 + 4\, v\cdot\phi'(\phi'^2)\right] + \frac{v^2\mu^2}{2} - \frac{\lambda}{8}(v^2)$$

but $v^2 = \frac{2\mu^2}{\lambda} \Rightarrow$ the linear term vanishes, as it must — we do not know how to handle a linear term — it's a signal for SSB.

and $\mu^2 - \frac{\lambda v^2}{2} = \mu^2 - \mu^2 = 0$

Now $v^i = \delta^{i4} v$ and $\frac{\lambda v^2}{2} = \mu^2$

So finally

$$\boxed{\mathcal{L} = \frac{1}{2}\partial_\mu \phi'^i \partial^\mu \phi'^i - \frac{(2\mu^2)}{2}\phi_4'^2 - \frac{\lambda}{8}\left[(\phi'^2)^2 + 4v\,\phi'_4\,\phi'^2\right] + \frac{v^4\lambda}{4}}$$

Introducing the field notation

$$\phi^{i\prime} \equiv \pi^i \qquad i = 1,2,3$$

$$\phi^{4\prime} \equiv \sigma$$

The Lagrangian is (with $i = 1,2,3$ now!)

$$\boxed{\begin{aligned} \mathcal{L} = {} & \tfrac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma + \tfrac{1}{2}\partial_\mu\pi^i\,\partial^\mu\pi^i \\ & - \frac{(2\mu^2)}{2}\sigma^2 - \frac{\lambda}{8}\left[(\sigma^2+\pi^2)^2 + 4v\sigma(\sigma^2+\pi^2)\right] \\ & + \frac{v^4\lambda}{4} \leftarrow \text{ignore constant.} \end{aligned}}$$

Hence we see that

1) Field that gets VEV is massive
this is σ-field with $\text{mass}^2 = 2\mu^2$
(The Higgs Boson)
2) Fields that get no VEV and transform into a constant plus fields are <u>massless</u> - called <u>Goldstone Bosons</u> -

Goldstone Theorem: Whenever a continuous global symmetry is spontaneously broken there will be associated with it massless scalar particles — called Goldstone bosons. Their # is equal to the # of broken symmetry generators

Here we have broke the O(4) symmetry by choosing a preferred non-zero VEV for the 4-direction ϕ^4 — thus only O(3) subgroup remains unbroken — O(4) has $\frac{4\cdot 3}{2}=6$ X's or generators

O(3) has $\frac{3\cdot 2}{2}=3$ generators

hence we have broken 3 symmetries hence 3 Goldstone Bosons arise the π^i, $i=1,2,3$.

3) The Lagrangian is invariant still but now in the transformation law $\phi^4 = \phi'^4 + v = \sigma + v$: thus the O(4) WI become what is called the Spontaneously broken O(4) to O(3) WI

$\delta_Q^{ij} Z[J] = 0$ still but now

$$\delta_Q^{ij}\phi^k = -i(T^{ij})_{kl}\phi^l$$
$$= -i(T^{ij})_{kl}(\phi'^l + \delta^{l4} v)$$
$$= \delta^j_k \phi^i - \delta^i_k \phi^j .$$

$$\boxed{\delta_Q^{ij}\phi^k = v(\delta^{i4}\delta^j_k - \delta^{j4}\delta^i_k) - i(T^{ij})_{kl}\phi'^l}$$

So if $k=4$

$$\delta_Q^{ij}\sigma = v(\delta^{i4}\delta^{j4} - \delta^{j4}\delta^{i4}) - i(T^{ij})_{4l}\phi'^l$$
$$= -i(T^{ij})_{4l}\phi'^l \quad \text{no inhomogeneous term}$$

for $k=1,2,3$

$$\delta_Q^{ij}\pi^k = v\underbrace{(\delta^{i4}\delta^{jk} - \delta^{j4}\delta^{ik})}_{\neq 0} - i(T^{ij})_{kl}\phi'^l$$

for $i=4$
$j=1,2,3.$

Along these lines note that the Noether current now becomes

$$J^{\mu}_{ij} = \phi^i \overleftrightarrow{\partial^\mu} \phi^j$$
$$= (\phi'^i + \delta^{i4} v) \overleftrightarrow{\partial^\mu} (\phi'^j + \delta^{j4} v)$$
$$= \phi'^i \overleftrightarrow{\partial^\mu} \phi'^j + v(\delta^{i4} \partial^\mu \phi'^j - \delta^{j4} \partial^\mu \phi'^i)$$

The current contains a term linear in the fields.

π-f

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4) As seen above it is messy to keep writing all the transformations and currents in O(4) notation — it is more convenient to express the symmetries in terms of broken and unbroken transformations & currents. Since the unbroken symmetry is an O(3) = SU(2) symmetry it is useful to write the transformations in the SU(2) vector and axial vector coset space form and in SU(2)$_L$ × SU(2)$_R$ to pick out the broken and unbroken generators more readily and then to couple to chiral fermions ~~more~~ easily.

Recall the O(4) algebra

$$[Q^{ij}, Q^{kl}] = i[\delta^{ik} Q^{jl} + \delta^{jl} Q^{ik} - \delta^{il} Q^{jk} - \delta^{jk} Q^{il}].$$

We would like to clarify the $SU(2)_L \times SU(2)_R$ structure of this group — of course there are many ways to identify the SU(2) subgroups. A particularly useful way — since we have given ϕ^4 the vacuum value — is to ask what generators are broken by it and what SU(2) is preserved by it. By choosing $\langle \phi_4 \rangle = v$ we have singled out the 4-direction and hence the broken and unbroken subgroups — Consider writing $[Q^{ij}, \phi^l]$ for the $\sigma, \vec{\pi}$ fields (we can proceed like Lorentz transformations — $\mu=0$ is like $i=4$ now!!)

(i.e. Rotations in 4-i plane broken, Rotations about 4 axis unbroken)

$$[Q^{ij}, \phi^l] = -(T^{ij})_{lm} \phi^m = i(\delta^i_l \delta^j_m - \delta^i_m \delta^j_l)\phi^m = i(\delta^i_l \phi^j - \delta^j_l \phi^i)$$

So

$$[Q^{ij}, \sigma] = i[\delta^{i4} \phi^j - \delta^{j4} \phi^i]$$

$$\Rightarrow [Q^{4j}, \sigma] = i\phi^j \qquad j = 1, 2, 3$$

$$\boxed{\begin{array}{l}\text{So for } i=1,2,3 \text{ we have} \\ \qquad [Q^{4i},\sigma]=i\pi^i \\ \text{and for } i,j=1,2,3 \\ \qquad [Q^{ij},\sigma]=0\end{array}}$$

Next consider $l=1,2,3$

$$[Q^{ij},\pi^l]=i(\delta^{il}\phi^j-\delta^{jl}\phi^i)$$

So $i=4$, $j=1,2,3$ $\Rightarrow$

$$\boxed{[Q^{4j},\pi^l]=-i\delta^{jl}(\sigma+v)}$$

and $i,j=1,2,3$ $\Rightarrow$

$$[Q^{ij},\pi^l]=i(\delta^{il}\pi^j-\delta^{jl}\pi^i)$$

$$=i[\delta^{il}\delta^{jk}-\delta^{jl}\delta^{ik}]\pi^k$$

$$\boxed{[Q^{ij},\pi^l]=i\epsilon^{ijm}\epsilon^{lkm}\pi^k}$$

Now we see that

$$\langle 0|[Q^{4j},\pi^{l}]|0\rangle = -i\delta^{jl} v \neq 0$$

$\Rightarrow$ $Q^{4j}|0\rangle \neq 0$ hence $|0\rangle$

is not invariant under the Q^{4i} transformations the Q^{4j} are the 3 broken symmetry generators while $Q^{ij}|0\rangle = 0$ for $i,j=1,2,3$ are the 3 unbroken symmetries

We will see that the Q^{ij} $(i,j=1,2,3)$ form a $SU(2)$ subgroup of $O(4)$.

Towards this end let's define

$$\boxed{\begin{aligned} &Q_A^i \left(= Q_5^i\right) \equiv Q^{4i} \\ &Q_V^i \equiv \tfrac{1}{2}\epsilon^{ijk} Q^{jk} \end{aligned}}$$

$i,j,k, = 1,2,3$ from here on

So

$$\epsilon^{lmi} Q_V^i = \tfrac{1}{2}\epsilon^{lmi}\epsilon^{jki} Q^{jk}$$

$$= \tfrac{1}{2}[\delta^{lj}\delta^{mk} - \delta^{lk}\delta^{mj}] Q^{jk}$$

$$= \tfrac{1}{2}[Q^{lm} - Q^{ml}] = Q^{lm}.$$

So $$\boxed{Q^{ij} = \epsilon^{ijk} Q_V^k}$$

Hence the field transformations become

$$\boxed{\begin{aligned} [Q_A^i, \sigma] &= i\pi^i \equiv -i\delta^i_{Q_A}\sigma \\ [Q_A^i, \pi^j] &= -i\delta^{ij}(\sigma+v) \equiv -i\delta^i_{Q_A}\pi^j \\ [Q_V^i, \sigma] &= 0 \equiv -i\delta^i_{Q_V}\sigma \\ [Q_V^i, \pi^j] &= i\epsilon^{ijk}\pi^k \equiv -i\delta^i_{Q_V}\pi^j \end{aligned}}$$

Now we could return to the Q^{ij} algebra and explicitly write it in terms of Q_A^i, Q_V^i just like we did in Relativity for J^i, K^i and the $M^{\mu\nu}$ algebra — instead we can more quickly use the above vector representation of O(4) to find the Q_A^i, Q_V^i commutators.

Note σ is a Q_V^i singlet — totally invariant

And π^i is a Q_V^i vector — it is in the SU(2) adjoint rep. of Q_V^i

$$[Q_V^i, \pi^j] \equiv -(L^i)_{jk}\,\pi^k$$

$$(L^i)_{jk} = -i\epsilon^{ijk} = +i\epsilon_{jik}$$

Recall this is the SU(2) vector or adjoint representation

$$[L^i, L^j] = i\epsilon^{ijk} L^k \qquad !!$$

So we have that

$$\boxed{[Q_V^i, Q_V^j] = i\epsilon^{ijk} Q_V^k}$$

The Q_V^i's are the generators for a (vector) SU(2) subgroup of O(4).

Further we find that

$$[[Q_V^i, Q_A^j], \sigma] = [Q_V^i, [Q_A^j, \sigma]] - [Q_A^j, \cancelto{0}{[Q_V^i, \sigma]}]$$

$$= i[Q_V^i, \pi^j] = -\epsilon^{ijk}\pi^k$$

$$= +i\epsilon^{ijk}[Q_A^k, \sigma]$$

$$\Rightarrow \boxed{[Q_V^i, Q_A^j] = +i\epsilon^{ijk} Q_A^k}$$

$$= -(L^i)_{jk} Q_A^k$$

The Q_A^i charges are vectors of $SU(2)_V$.

We can check that

$$[[Q_V^i, Q_A^j], \pi^k] = i\epsilon^{ijl}[Q_A^l, \pi^k]$$

also.

Finally

$$[[Q_A^i, Q_A^j], \sigma] = [Q_A^i, [Q_A^j, \sigma]] - [Q_A^j, [Q_A^i, \sigma]]$$

$$= i[Q_A^i, \pi^j] - i[Q_A^j, \pi^i]$$

$$= \delta^{ij}(\sigma+v) - \delta^{ij}(\sigma+v) = 0$$

$$= C_{ijk} [Q_V^k, \sigma]$$

arb. const. so far.

To find C_{ijk} we use the final commutator

$$[[Q_A^i, Q_A^j], \pi^k] = [Q_A^i, [Q_A^j, \pi^k]] - [Q_A^j, [Q_A^i, \pi^k]]$$

$$= -i\delta^{jk}[Q_A^i, \sigma] + i\delta^{ik}[Q_A^j, \sigma]$$

$$= -i\delta^{jk}\, i\pi^i + i\delta^{ik}\, i\pi^j$$

$$= (\delta^{jk}\delta^{il} - \delta^{ik}\delta^{jl})\pi^l$$

$$= -\epsilon^{ijm}\epsilon^{mkl}\pi^l = i\epsilon^{ijm}\, i\epsilon^{mkl}\pi^l$$

$$= i\epsilon^{ijm}[Q_V^m, \pi^k]$$

$$\Rightarrow \boxed{[Q_A^i, Q_A^j] = i\epsilon^{ijk} Q_V^k}$$

Thus we have identified the unbroken $SU(2)_V$ subgroup of $O(4)$ — it is generated by the Q_V^i. Further we have written the algebra of $O(4)$ in terms of these $SU(2)_V$ unbroken generators and the 3 broken generators of the coset space $O(4)/SU(2)_V$.

The O(4) algebra becomes

$$\boxed{\begin{aligned} [Q_V^i, Q_V^j] &= +i\epsilon_{ijk} Q_V^k \\ [Q_V^i, Q_A^j] &= +i\epsilon_{ijk} Q_A^k \\ [Q_A^i, Q_A^j] &= +i\epsilon_{ijk} Q_V^k \end{aligned}}$$

We can define the Parity transformation (involutive automorphism) on the algebra by

$$P^{-1} Q_V^i P = +Q_V^i$$

$$P^{-1} Q_A^i P = -Q_A^i$$

Thus we call the Q_V^i generated subgroup the SU(2)-vector subgroup — hence the subscript V, $SU(2)_V$.

and the coset space generators Q_A^i are axial vector charges hence the subscript A. Hence we have when applying the Parity transformation to the field commutation relations that

$$\mathcal{P}^{-1}\sigma(x^0,\vec{x})\mathcal{P} = +\sigma(x^0,-\vec{x})$$

$$\mathcal{P}^{-1}\pi^i(x^0,\vec{x})\mathcal{P} = -\pi^i(x^0,-\vec{x})$$

That is σ is a scalar field and π^i are pseudoscalar fields.

Next we can consider Noether's theorem and the Noether currents for vector and axial vector transformations. We can apply Noether's theorem directly to the σ-π Lagrangian or we can transform the $O(4)$ currents into their vector and axial vector components as we did with the charges.

Applying Noether's theorem to $\mathcal{L}(\sigma,\pi)$ we have

$$J_V^{i\mu} \equiv \frac{\partial\mathcal{L}}{\partial\partial_\mu\pi^j}\delta^i_{QV}\pi^j + \frac{\partial\mathcal{L}}{\partial\partial_\mu\sigma}\delta^i_{QV}\sigma$$

$$J_V^{i\mu} = \partial^\mu \pi^j (-\epsilon^{ijk}\pi^k)$$

$$= \epsilon^{ijk}\pi^j \partial^\mu \pi^k = (\vec{\pi}\times\partial^\mu\vec{\pi})^i$$

$$\boxed{J_V^{i\mu} = \tfrac{1}{2}\epsilon^{ijk}\pi^j \overleftrightarrow{\partial^\mu} \pi^k = \tfrac{1}{2}(\vec{\pi}\times\overleftrightarrow{\partial^\mu}\vec{\pi})^i}$$

and

$$J_A^{i\mu} \equiv \frac{\partial \mathcal{L}}{\partial \partial_\mu \pi^j}\delta^i_{Q_A}\pi^j + \frac{\partial \mathcal{L}}{\partial \partial_\mu \sigma}\delta^i_{Q_A}\sigma$$

$$= \partial^\mu \pi^j (\delta^{ij}(\sigma+v)) + \partial^\mu\sigma(-\pi^i)$$

$$\boxed{J_A^{i\mu} = (\sigma+v)\overleftrightarrow{\partial^\mu}\pi^i}$$

Now Noether's theorem, since $\delta^i_{Q_{V,A}}\mathcal{L}=0$, $\Rightarrow$

1) $\partial_\mu J_V^{i\mu}(x)\, Z[J] = -i\,\delta^i_{Q_V}(x)\, Z[J]$

2) $\partial_\mu J_A^{i\mu}(x)\, Z[J] = -i\,\delta^i_{Q_A}(x)\, Z[J]$

where

$$\delta^i_{Q_V}(x) \equiv J^j_\pi(x)\left(-\epsilon^{ijk}\frac{\delta}{\delta J^k_\pi(x)}\right)$$

$$\delta^i_{Q_A}(x) \equiv \left\{ J^j_\pi(x)\left(i\delta^{ij}v + \delta^{ij}\frac{\delta}{\delta J_\sigma(x)}\right) + J_\sigma(x)\left(-\frac{\delta}{\delta J^i_\pi(x)}\right)\right\}$$

and the vector and axial vector WI operators are given by

$$\delta^i_{Q_V} \equiv \int d^4x\, \delta^i_{Q_V}(x)$$

$$\delta^i_{Q_A} \equiv \int d^4x\, \delta^i_{Q_A}(x) \text{ and}$$

$$Z[J] = \langle 0|T e^{i\int dx\,[J^i_\pi \pi^i + J_\sigma \sigma]}|0\rangle .$$

Note that these are the same form as the O(4) WI operators except that we have $(\sigma + v)$ in the operator — it is the spontaneously broken O(4) to $SU(2)_V$ WI operators.

We now would like to integrate Noether's theorem over x to obtain the SBWI

<u>But</u> here we run into a subtlety due to the <u>masslessness</u> of the pion

Since
$$J^{i\mu}_A = v\partial^\mu \pi^i + \sigma \overleftrightarrow{\partial^\mu} \pi^i$$

$$\partial_\mu J^{i\mu}_A = v\partial^2 \pi^i + \sigma \partial^2 \pi^i - \partial^2 \sigma \pi^i$$

Now if we integrate and throw away surface terms $\int d^4x\, \partial_\mu J^{i\mu}_A Z = 0$

we have a contradiction since π^i is massless $\partial^2 \pi^i \sim p^2 \pi^i$ amputates the pion pole in any Green function

hence one is left with something non-zero; i.e. the $\frac{i\delta^{ij}}{p^2}$ cancels the p^2 to leave a residue!!

Technically we can avoid this problem of the $p \to 0$ limit by giving the pion a mass - then letting $p \to 0$ first and at the end of everything take the mass to zero. The mass can be added by adding an explicit breaking term to the Lagrangian $\mathcal{L} \to \mathcal{L} + \mathcal{L}_{break}$

$\mathcal{L}_{break} = c\sigma$ so that

$$\delta^i_{Q_V} \mathcal{L} = 0 \quad SU(2)_V \text{ is unbroken but}$$
$$\delta^i_{Q_A} \mathcal{L} = -c\pi^i \neq 0.$$

This requires a shift in the vacuum expectation value - a new minimum and π^i is no longer massless i.e.

$$\mu^2 - \frac{\lambda v^2}{2} + \frac{c}{v} = 0 \Rightarrow \mu^2 - \frac{\lambda v^2}{2} = -\frac{c}{v} \neq 0.$$

Hence the π mass term becomes

$$-\frac{1}{2}\left(\frac{c}{v}\right)\pi^i\pi^i$$

and the σ-mass term

$$-\frac{1}{2}\left(2\mu^2+\left(\frac{c}{v}\right)\right)\sigma^2 .$$

We can throw away total divergences to obtain ~~from~~ Noether theorem the action principle

$$\partial_\mu J^{i\mu}_{A}(x)\,Z[J] = -i\,\delta^i_{Q_A}(x)\,Z[J] + \delta^i_{Q_A}\mathcal{L}(x)\,Z[J].$$

$\Rightarrow$

1) $$\delta^i_{Q_V} Z[J] = -i\int d^4x\, \delta^i_{Q_V}\mathcal{L}(x)\,Z[J] = 0$$

2) $$\delta^i_{Q_A} Z[J] = -i\int d^4x\, \delta^i_{Q_A}\mathcal{L}(x)\,Z[J] = -ic\int d^4x\, \frac{\delta}{i\,\delta J^i_\pi(x)} Z[J].$$

We can then define the Legendre transform to the effective action for $c \neq 0$ and study the broken WI for it. More on this later.

Hence we can apply these WI to the general Green function to obtain.

1) For the unbroken subgroup $SU(2)_V$

$$0 = \sum_{a=1}^{n} \langle 0 | T\, \sigma(x_1) \cdots \sigma(x_m)\, \pi^{i_1}(y_1) \cdots \left(-\epsilon^{i_a j}\, \pi^j(y_a)\right) \cdots \pi^{i_n}(y_n) | 0 \rangle$$

i.e. $\langle 0 | T\, \sigma \cdots \sigma\, \vec{\pi} \cdots \vec{\pi} | 0 \rangle$ is invariant under $SU(2)_V$ rotations with σ a scalar and $\vec{\pi}$ a $SU(2)_V$ vector.

2) For the broken generators of the coset space $O(4)/SU(2)_V$.

$$\sum_{a=1}^{m} \langle 0 | T\, \sigma(x_1) \cdots \left(-\pi^i(x_a)\right) \cdots \sigma(x_m)\, \pi^{i_1}(y_1) \cdots \pi^{i_n}(y_n) | 0 \rangle$$

$$+ \sum_{b=1}^{n} \langle 0 | T\, \sigma(x_1) \cdots \sigma(x_m)\, \pi^{i_1}(y_1) \cdots \left[i\delta^{i i_b} v + \delta^{i i_b} \sigma(y_b) \right] \cdots \pi^{i_n}(y_n) | 0 \rangle$$

$$= -c \int d^4x\, \langle 0 | T\, \pi^i(x)\, \sigma(x_1) \cdots \sigma(x_m)\, \pi^{i_1}(y_1) \cdots \pi^{i_n}(y_n) | 0 \rangle$$

Note that the $\overset{SB}{}$ Axial vector WI's are quite different due to the VEV. We have related (m,n) point functions to $(m-1,n)$ point and for $c\neq 0$ $(m,n+1)$ point functions. For example we have (in cryptic notation)

$$\Big(\delta^{i}_{QA}\,\langle 0|T\,\sigma(x)\,\pi^{j}(y)\,\pi^{k}(z)\overset{\pi^{l}(w)}{}|0\rangle$$

$$=\Big)\langle 0|T(-\pi^{i}(x))\,\pi^{j}(y)\,\pi^{k}(z)\overset{\pi^{l}(w)}{\vee}|0\rangle$$

$$+\langle 0|T\,\sigma(x)\big(\delta^{ij}(\sigma(y)+v)\big)\,\pi^{k}(z)\overset{\pi^{l}(w)}{\vee}|0\rangle$$

$$+\langle 0|T\,\sigma(x)\,\pi^{j}(y)\big(\delta^{ik}(\sigma(z)+v)\big)\overset{\pi^{l}(w)}{\vee}|0\rangle$$

$$+\langle 0|T\,\sigma(x)\,\pi^{j}(y)\,\pi^{k}(z)\big(\delta^{il}(\sigma(w)+v)\big)|0\rangle$$

$$=-c\int d^4u\,\langle 0|T\,\pi^{i}(u)\,\sigma(x)\,\pi^{j}(y)\,\pi^{k}(z)\,\pi^{l}(w)|0\rangle$$

So we have $\Rightarrow$

$$\langle 0|T\,\pi^{i}(x)\,\pi^{j}(y)\,\pi^{k}(z)\,\pi^{l}(w)|0\rangle$$

$$=\langle 0|T\,\sigma(x)\,\delta^{ij}\sigma(y)\,\pi^{k}(z)\,\pi^{l}(w)|0\rangle$$

$$+\langle 0|T\,\sigma(x)\,\delta^{ik}\sigma(z)\,\pi^{j}(y)\,\pi^{l}(w)|0\rangle$$

$$+\langle 0|T\,\sigma(x)\,\delta^{il}\sigma(w)\,\pi^{j}(y)\,\pi^{k}(z)|0\rangle$$

$$+ c\int d^4u \langle 0|T\, \pi^i(u)\,\sigma(x)\,\pi^j(y)\,\pi^k(z)\,\pi^l(w)|0\rangle$$

$$+ \delta^{ij} v \langle 0|T\, \sigma(x)\,\pi^k(z)\,\pi^l(w)|0\rangle$$

$$+ v\delta^{ik} \langle 0|T\, \sigma(x)\,\pi^j(y)\,\pi^l(w)|0\rangle$$

$$+ v\delta^{il} \langle 0|T\, \sigma(x)\,\pi^j(y)\,\pi^k(z)|0\rangle$$

Hence the SBWI relates the 4π, $1\text{-}\sigma$ 5-pt function, 4π $2\sigma\text{-}2\pi$ 4 pt. function to the $\sigma\text{-}2\pi$ 3 pt. function (We can calculate this in lowest order perturbation theory to verify that it is true, messy & easier to work with 1-PI functions).

Finally we would like to be able to simply couple fermions to such a model. In the case that the fermions transform according to $SU(2)_L \times SU(2)_R$ as in the Standard model we must exhibit the $SU(2)_L \times SU(2)_R$ transformation properties of π & σ.

Again appealing to our $SL(2,C) \sim SO(3,1)$ analysis in relativity the left-right structure of $O(4)$ is seen by introducing the left, L^i and right, R^i charges or a linear compo's of Q_V^i & Q_A^i just like $N_i, N_i^\dagger$ were related to J^i & K^i.

Define
$$\boxed{\begin{aligned} L^i &\equiv \tfrac{1}{2}(Q_V^i - Q_A^i) \\ R^i &\equiv \tfrac{1}{2}(Q_V^i + Q_A^i) \end{aligned}}$$

$\Rightarrow$

$$Q_V^i = R^i + L^i$$

$$Q_A^i = R^i - L^i$$

Hence the Q_V, Q_A algebra decomposes into

$$[L^i, L^j] = +i\epsilon^{ijk} L^k$$

$$[R^i, R^j] = +i\epsilon^{ijk} R^k$$

$$[R^i, L^j] = 0$$

And thus we see that the L^i

for $SU(2)_L$ and the R^i ~~the~~ $SU(2)_R$.

The σ, π^i transformations become

$$\boxed{\begin{aligned}
[L^i,\sigma] &= -\tfrac{i}{2}\pi^i \equiv -i\delta_L^i\sigma \\
[L^i,\pi^j] &= \tfrac{i}{2}\epsilon^{ijk}\pi^k + \tfrac{i}{2}\delta^{ij}(\sigma+v) \equiv -i\delta_L^i\pi^j \\
[R^i,\sigma] &= +\tfrac{i}{2}\pi^i \equiv -i\delta_R^i\sigma \\
[R^i,\pi^j] &= \tfrac{i}{2}\epsilon^{ijk}\pi^k - \tfrac{i}{2}\delta^{ij}(\sigma+v) \equiv -i\delta_R^i\pi^j
\end{aligned}}$$

Recall that we have a 1-1 correspondence between 2×2 matrices (pseudo-Hermitian) and points in $O(4)$ through

$$\Sigma_1 \equiv (v+\sigma)\mathbb{1} + i\pi^i\,\tau^i$$

with τ^i = Pauli matrices $\tau^1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\tau^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$

$\tau^3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$$\Sigma_1 = \begin{bmatrix}(\sigma+v+i\pi^3) & (\pi^2+i\pi^1)\\ (-\pi^2+i\pi^1) & (\sigma+v-i\pi^3)\end{bmatrix}$$

The $\det \Sigma_4' = (\sigma+v)^2 + \pi^i\pi^i$ is an $O(4)$ invariant. Further we have that under $SU(2)_L \times SU(2)_R$ the matrix transforms as

$$[R^i, \Sigma_4'] = \frac{i}{2}\pi^i \mathbb{1} + i\tau^j\left(\frac{i}{2}\epsilon^{ijk}\pi^k - \frac{i}{2}\delta^{ij}(\sigma+v)\right)$$

$$= +(\sigma+v)\frac{\tau^i}{2} + i\frac{\pi^k}{2}\left(\delta^{ik} + i\epsilon^{ijk}\tau^j\right)$$

Now recall that

$$\tau^i\tau^k = \delta^{ik} + i\epsilon^{ikj}\tau^j$$
$$\tau^k\tau^i = \delta^{ik} - i\epsilon^{ikj}\tau^j$$

So

$$[R^i, \Sigma_4'] = +(\sigma+v)\frac{\tau^i}{2} + \frac{i}{2}\pi^k\tau^k\tau^i$$

$$= \left((\sigma+v)\mathbb{1} + i\tau^k\pi^k\right)\frac{\tau^i}{2}$$

$$\boxed{[R^i, \Sigma_4'] = \Sigma_4'\frac{\tau^i}{2}} \equiv -i\delta_R^i\Sigma_4'$$

And similarly

$$[L^i, \Sigma_1] = -\frac{i}{2}\pi^i + i\tau^j\left(\frac{i}{2}\epsilon^{ijk}\pi^k + \frac{i}{2}\delta^{ij}(\sigma+\nu)\right)$$

$$= -\frac{\tau^i}{2}(\sigma+\nu) - \frac{i}{2}\pi^k(\delta^{ik} - i\epsilon^{ijk}\tau^j)$$

$$= -\frac{\tau^i}{2}\left[(\sigma+\nu)\mathbb{1} + i\pi^k\tau^k\right]$$

$$\boxed{[L^i, \Sigma_1] = -\frac{\tau^i}{2}\Sigma_1} \equiv -i\delta_L^i\Sigma_1$$

Hence we find that

$$[\theta_L^i L^i + \theta_R^i R^i, \Sigma_1] = -\frac{\vec{\theta}_L\cdot\vec{\tau}}{2}\Sigma_1 + \Sigma_1\frac{\vec{\theta}_R\cdot\vec{\tau}}{2}$$

or

$$S(\theta) \equiv e^{-i\vec{\theta}\cdot\frac{\vec{\tau}}{2}} S_0$$

$$\boxed{\mathcal{U}^{-1}(\theta_L,\theta_R)\,\Sigma_1\,\mathcal{U}(\theta_L,\theta_R) = S(\theta_L)\,\Sigma_1\,S^{\dagger}(\theta_R)}$$

with $\mathcal{U}(\theta_L,\theta_R) = e^{-i\vec{\theta}_L\cdot\vec{L} - i\vec{\theta}_R\cdot\vec{R}}$

Now we can define fields that transform as spinors under $SU(2)_L \times SU(2)_R$; that is as we see Σ_i^{-1} is a $(\frac{1}{2},\frac{1}{2})$ field under $SU(2)_L \times SU(2)_R$, we can define left handed fermions to transform as $(\frac{1}{2},0)$ under $SU(2)_L \times SU(2)_R$

$$\mathcal{U}^{-1}(\theta_L,\theta_R)\,\Psi_L\,\mathcal{U}(\theta_L,\theta_R) = S(\theta_L)\Psi_L$$

$$\Rightarrow \boxed{[L^i, \Psi_L] = -\frac{\tau^i}{2}\Psi_L \equiv -i\delta_L^i \Psi_L}$$

i.e. $\Psi_L^j = \frac{1}{2}(1-\gamma_5)\Psi^j$ (don't confuse Dirac index with $SU(2)_{L,R}$ index j!!)

and $\boxed{[R^i, \Psi_L] = 0 \equiv -i\delta_R^i \Psi_L}$

Thus $\overline{(\Psi_L)} = \overline{\Psi}_L \equiv \Psi_L^\dagger \gamma^0 = \frac{1}{2}\Psi^\dagger(1-\gamma_5)\gamma^0$

$$= \frac{1}{2}\overline{\Psi}(1+\gamma_5) = (\overline{\Psi})_R$$

transforms as

$$\tau^{i\dagger} = \tau^i$$

$$[L^i, \Psi_L^\dagger] = +\Psi_L^\dagger \frac{\tau^i}{2}$$

$$\Rightarrow \boxed{[L^i, \overline{\Psi}_L] = +\overline{\Psi}_L \frac{\tau^i}{2}} \equiv -i\delta_L^i \overline{\Psi}_L$$

Hence $\overline{\Psi}_L \gamma \Psi_L$ is invariant under $SU(2)_L \times SU(2)_R$,

$$\delta^i_{\substack{L\\R}} \overline{\Psi}_L \gamma \Psi_L = 0 .$$

Similarly we can introduce right handed fermions transforming as the $SU(2)_R$ doublet i.e. $(0, \frac{1}{2})$

$$\mathcal{U}^{-1}(\theta_L, \theta_R)\, \Psi_R\, \mathcal{U}(\theta_L, \theta_R) \equiv S(\theta_R)\, \Psi_R$$

$$\Rightarrow \boxed{[R^i, \Psi_R] = -\frac{\tau^i}{2} \Psi_R \equiv -i\delta_R^i \Psi_R}$$

and $\boxed{[L^i, \Psi_R] = 0 \equiv -i\delta^i_L \Psi_R}$

Hence $\overline{\Psi}_R \equiv \Psi_R^{\dagger} \gamma^0$ transforms as a right-handed $\overline{2}$,

$$\boxed{[R^i, \overline{\Psi}_R] = +\overline{\Psi}_R \frac{\tau^i}{2} \equiv -i\delta^i_R \overline{\Psi}_R}$$

~~Similar~~ Hence $\bar{\Psi}_R \gamma \Psi_R$ is an $SU(2)_L \times SU(2)_R$ invariant.

$$\delta^i_{\frac{L}{R}} \bar{\Psi}_R \gamma \Psi_R = 0 .$$

Of course $\bar{\Psi}_L \Psi_R$, $\bar{\Psi}_R \Psi_L$ are not invariants — no invariant mass terms can be written down alone.

Now if Ψ_L is $(\frac{1}{2},0)$; $\Psi_R (0,\frac{1}{2})$ and Σ^i $(\frac{1}{2},\bar{\frac{1}{2}})$ we can make a singlet by

$$\bar{\Psi}_L \Sigma^i \Psi_R$$

i.e.

$$\mathcal{U}^{-1}(\theta_L,\theta_R)\,\overline{\Psi}_L \vec{\Sigma}\,\Psi_R\,\mathcal{U}(\theta_L,\theta_R)$$

$$= \overline{\Psi}_L S^{\dagger}(\theta_L) S(\theta_L) \vec{\Sigma} \;\; S^{\dagger}(\theta_R) S_R(\theta_R)\Psi_R$$

$$= \overline{\Psi}_L \vec{\Sigma}\,\Psi_R$$

Hence adding to this the complex conjugate we have the most general $SU(2)_L \times SU(2)_R$ Lagrangian

$$\mathcal{L} = \mathcal{L}_\sigma + \mathcal{L}_F + \mathcal{L}_{yuk}$$

$$\mathcal{L}_\sigma = \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma + \tfrac{1}{2}\partial_\mu\vec{\pi}\cdot\partial^\mu\vec{\pi} - \frac{(2\mu^2)}{2}\sigma^2$$

$$-\frac{\lambda}{8}\left[(\sigma^2+\pi^2)^2 + 4N\sigma(\sigma^2+\pi^2)\right] + c\sigma$$

$$\mathcal{L}_F = \tfrac{i}{2}\overline{\Psi}_L \overleftrightarrow{\not\partial}\Psi_L + \tfrac{i}{2}\overline{\Psi}_R\overleftrightarrow{\not\partial}\Psi_R = \tfrac{i}{2}\overline{\Psi}\overleftrightarrow{\not\partial}\Psi$$

$$\mathcal{L}_{yuk} = -g\,\overline{\Psi}_L\left((\sigma+N)\mathbb{1} + i\vec{\pi}\cdot\vec{\tau}\right)\Psi_R$$

$$-g\,\overline{\Psi}_R\left((\sigma+N)\mathbb{1} - i\vec{\pi}\cdot\vec{\tau}\right)\Psi_L$$

where we have used $\Psi_L + \Psi_R = \Psi$

Now

$$\bar{\Psi}_L \Psi_R + \bar{\Psi}_R \Psi_L = \bar{\Psi}\Psi$$

$$= \Psi^\dagger \gamma^0 \left(\tfrac{1}{2}\right)(1+\gamma_5)\Psi + \Psi^\dagger \gamma^0 \tfrac{1}{2}(1-\gamma_5)\Psi = \bar{\Psi}\Psi \checkmark$$

and

$$\bar{\Psi}_L \Psi_R - \bar{\Psi}_R \Psi_L = \bar{\Psi}\,\tfrac{1}{2}(1+\gamma_5)\Psi$$
$$- \bar{\Psi}\,\tfrac{1}{2}(1-\gamma_5)\Psi$$
$$= \bar{\Psi}\gamma_5\Psi$$

So

$$\boxed{\mathcal{L}_{Yuk} = -g\,\bar{\Psi}\left(\sigma + v + i\gamma_5 \vec{\pi}\cdot\vec{\tau}\right)\Psi}$$

Notice that the fermions are given a mass by the vac. exp. value

$m_f = gv$.

Again one can construct the currents and Noether's theorem etc. — this is left as an exercise for the reader.

III.E.3) Local Internal Symmetries — Gauge Invariance Revisited.

Suppose we have a theory which is described by a Lagrangian with a global internal symmetry.

$\mathcal{L} = \mathcal{L}(\varphi, \partial_\mu \varphi)$ such that

$$\delta_Q^a \mathcal{L}(\varphi, \partial_\mu \varphi) = 0 \text{ and}$$

$$\delta_Q^a \varphi^\alpha(x) = -i(T^a)_{\alpha\beta}\varphi^\beta(x)$$

That is $\mathcal{L}$ is invariant under global rotations of φ^α through angles ω^a

$$\mathcal{U}^{-1}(\omega)\,\phi^\alpha(x)\,\mathcal{U}(\omega) = U(\omega)_{\alpha\beta}\,\phi^\beta(x)$$