III.F) Symmetries, Noether's Theorem, Schwinger's Quantum Action Principle and Ward Identities

Symmetries are divided into two major classes global (symmetries of the first kind) and local (symmetries of the second kind). Global symmetries involve the same transformation at every space-time point. For example the transformations of special relativity are transformations of the first kind since every space-time point is boosted by the same velocity or rotated through the same angle. Local symmetries involve transformations that differ at every space-time point. For example the gauge invariance of QED involves a phase transformation of the electron fields that is a function of space-time and its derivative is the inhomogeneous term of the photon's transformation law. As well we could imagine Lorentz transforming each space-time point differently. These local Lorentz symmetry transformations correspond to those of general relativity.

Hence we can always ask if a global symmetry can be made local. Theoretically this will involve the introduction of new forces ~~e.g.~~ General relativity implies the existence of gravitational forces while global Lorentz invariance did not. Of course whether Nature respects such symmetries is an experimental question.

Further we see that each class of transformations can be categorized as to whether they involve transformation of space-time or not. If not they are called internal symmetries, if they do they are called space-time symmetries. Hence we have 4 classes of symmetry transformations

1) Global space time symmetries
2) Global internal symmetries
3) Local internal symmetries
4) Local space-time symmetries.

Let's begin by recalling the global space-time symmetries of the Poincaré group.

1) The Poincaré invariance of the S operator was expressed by the fact that it commuted with the unitary operators implementing the Poincaré transformations in Hilbert space

$$S = U^{-1}(\Lambda,a)\, S\, U(\Lambda,a)$$

where

$$U(\Lambda,a) = e^{+ia_\mu P^\mu}\, e^{-\frac{i}{2}\omega_{\mu\nu}(\Lambda)M^{\mu\nu}}.$$

In particular we can apply this to our LSZ expression for S

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!}\int dx_1 \cdots dx_n \Big[Z^{-1/2} K_{x_1} \cdots Z^{-1/2} K_{x_n} \times$$

$$\times \langle 0| T\,\phi(x_1)\cdots\phi(x_n)|0\rangle \Big] :\phi_{in}(x_1)\cdots\phi_{in}(x_n):$$

$$= U^{-1}(\Lambda,a)\, S\, U(\Lambda,a)$$

$$= \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \ldots dx_n \Big[Z^{-1/2} K_{x_1} \cdots Z^{-1/2} K_{x_n} \times$$

$$\times \langle 0 | T \phi(x_1) \cdots \phi(x_n) | 0 \rangle \Big] : D(\Lambda) \phi_{in}(\Lambda^{-1}(x_1 - a)) \cdots$$

$$\cdots D(\Lambda) \phi_{in}(\Lambda^{-1}(x_n - a)) :$$

where we used the fact that ϕ_{in} belongs to a finite dimensional representation, $D(\Lambda)$, of the Lorentz group so that

$$U^{-1}(\Lambda, a)\, \phi_{in}(x)\, U(\Lambda, a) = D(\Lambda)\, \phi_{in}(\Lambda^{-1}(x - a))$$

We change variables $x_i' = [\Lambda^{-1}(x - a)]_i$

so that $x_i = (\Lambda x' + a)_i$ and $d^4 x_i = \det \Lambda \, d^4 x_i'$
$= d^4 x_i'$

and $\quad K_x = D K_{x'} D^{-1}$

(i.e.) 1) if ϕ_{in} is a scalar $D(\Lambda) = 1$ and $K_x = (\partial_x^2 + m^2)$
$= K_{x'}$

2) if ϕ_{in} is a spinor $D(\Lambda) = e^{-\frac{i}{2} \omega_{\mu\nu}(\Lambda) \frac{\sigma^{\mu\nu}}{2}}$

and $K_x = (i\not{\partial}_x - M) = D(\Lambda) K_{x'} D^{-1}(\Lambda)$

since
$$\partial^x_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\frac{\partial}{\partial x'^\nu} = (\Lambda^{-1})^\nu{}_\mu \partial^{x'}_\nu$$
$$= \Lambda_{\mu\nu}\partial^\nu_{x'}$$

So
$$\not{\partial}_x = \gamma^\mu \Lambda_{\mu\nu}\partial^\nu_{x'} \text{ but}$$
$$\Lambda^{\mu\nu}\gamma_\mu = D\gamma^\nu D^{-1}$$
(we called $D = L(\Lambda)$ last semester)

So
$$\not{\partial}_x = D\not{\partial}_{x'}D^{-1} \quad \checkmark.)$$

Hence we have
$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!}\int dx_1' \cdots dx_n' \left[Z^{-1/2}K_{x_1'} \cdots Z^{-1/2}K_{x_n'} \times \right.$$
$$\left. \times \langle 0 | T\, D^{-1}(\Lambda)\phi(\Lambda x_1' + a) \cdots D^{-1}(\Lambda)\phi(\Lambda x_n' + a) | 0 \rangle \right] \times$$
$$\times :\phi_{in}(x_1') \cdots \phi_{in}(x_n'):$$

relabeling $x_i' \to x_i$ we have

$$S = \sum_{n=0}^{\infty} \frac{i^n}{n!} \int dx_1 \cdots dx_n \Big[Z^{-1/2} K_{x_1} \cdots Z^{-1/2} K_{x_n} \times$$

$$\times \langle 0 | T\, D^{-1}(\Lambda)\phi(\Lambda x_1 + a) \cdots D^{-1}(\Lambda)\phi(\Lambda x_n + a) | 0 \rangle \Big] \times$$

$$\times :\phi_{in}(x_1) \cdots \phi_{in}(x_n):$$

Comparing to our original expression for S we have if S is to be Poincaré invariant then

$$\boxed{\begin{aligned} &\langle 0 | T\, \phi(x_1) \cdots \phi(x_n) | 0 \rangle \\ &\quad = \langle 0 | T\, D^{-1}(\Lambda)\phi(\Lambda x_1 + a) \cdots D^{-1}(\Lambda)\phi(\Lambda x_n + a) | 0 \rangle \end{aligned}}$$

This is just the expression of Lorentz and translation invariance of the time ordered functions. Hence it is sufficient that the time ordered functions are Poincaré invariant for the S operator to be. (sufficient because we can have Poincaré invariance up to contact terms — these do not contribute to S-matrix — of course we can define new T-operator

So these contact terms are absent)

Since the finite Poincaré transformations are built up from infinitesimal ones we can ask for invariance under infinitesimal Poincaré transformations

letting $D(\Lambda) \equiv e^{-\frac{i}{2}\omega_{\mu\nu}(\Lambda)\Sigma_i^{\mu\nu}}$

(i.e. scalars $\Sigma_i^{\mu\nu} = 0$
spinors $\Sigma_i^{\mu\nu} = \frac{\sigma^{\mu\nu}}{2}$
vectors $(\Sigma_i^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\beta\delta^\nu_\alpha)$, etc

see review of Lorentz group in 662 notes)

For infinitesimal transformations $\Lambda^{\mu\nu} = g^{\mu\nu} + \omega^{\mu\nu}$ and $a^\mu = a^\mu$ with $a^\mu, \omega^{\mu\nu}$ infinitesimal
We have $\omega^{\mu\nu}(\Lambda^{\mu\nu}) = \omega^{\mu\nu}(g+\omega) = \omega^{\mu\nu}$
$= -\omega^{\nu\mu}$

and $D(\Lambda) = 1 - \frac{i}{2}\omega_{\mu\nu}\Sigma_i^{\mu\nu}$ with

$U(\Lambda, a) = 1 + i a_\mu P^\mu - \frac{i}{2}\omega_{\mu\nu}M^{\mu\nu}$

and the ϕ (or ϕ_{in}) transformation law becomes

$$\mathcal{U}^{-1}(\Lambda,a)\,\phi(x)\,\mathcal{U}(\Lambda,a) = D(\Lambda)\,\phi(\Lambda^{-1}(x-a))$$

$$\phi(x) - \left[i a_\mu \mathcal{P}^\mu - \tfrac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}, \phi(x)\right] = \left(1 - \tfrac{i}{2}\omega_{\mu\nu}\Sigma_1^{\mu\nu}\right) \times$$
$$\times \Big[\phi(x) - a^\mu \partial_\mu \phi(x)$$
$$+ \tfrac{1}{2}\omega_{\mu\nu}(x^\mu\partial^\nu - x^\nu\partial^\mu)\phi(x) \Big]$$

$\Rightarrow$

$$\left[i a_\mu \mathcal{P}^\mu - \tfrac{i}{2}\omega_{\mu\nu}\mathcal{M}^{\mu\nu}, \phi(x)\right]$$
$$= + a^\mu \partial_\mu \phi(x) + \tfrac{i}{2}\omega_{\mu\nu}\left[\left(i(x^\mu\partial^\nu - x^\nu\partial^\mu) + \Sigma_1^{\mu\nu}\right)\phi(x)\right]$$

Hence we have as you will recall

$$\left[\mathcal{P}^\mu, \phi(x)\right] = -i\partial^\mu \phi(x) \equiv -i\delta_P^\mu \phi(x)$$
$$\left[\mathcal{M}^{\mu\nu}, \phi(x)\right] = -i\left[(x^\mu\partial^\nu - x^\nu\partial^\mu) - i\Sigma_1^{\mu\nu}\right]\phi(x)$$
$$\equiv -i\delta_M^{\mu\nu}\phi(x)$$

So we would like to apply the same type of expansion to the Green function of

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$=\langle 0|T\Big[\big(1+\tfrac{i}{2}\omega_{\mu\nu}\Sigma_1^{(\mu\nu)}\big)\big(1+a^\mu\partial_\mu-\tfrac{1}{2}\omega_{\mu\nu}(x_1^\mu\partial_1^\nu-x_1^\nu\partial_1^\mu)\big)\phi(x_1)\Big]$$

$$\times\cdots\times\Big[\big(1+\tfrac{i}{2}\omega_{\mu\nu}\Sigma_1^{(\mu\nu)}\big)\big(1+a^\mu\partial_{n\mu}-\tfrac{1}{2}\omega_{\mu\nu}(x_n^\mu\partial_n^\nu-x_n^\nu\partial_n^\mu)\big)\phi(x_n)\Big]$$

$$\times|0\rangle$$

expanding we have

$$=\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle$$

$$+\sum_{i=1}^{n}\langle 0|T\phi(x_1)\cdots\phi(x_{i-1})\times$$

$$\times\Big[a^\mu\partial_\mu^{x_i}-\tfrac{1}{2}\omega_{\mu\nu}\big(x_i^\mu\partial_i^\nu-x_i^\nu\partial_i^\mu-i\Sigma_1^{(\mu\nu)}\big)\Big]\phi(x_i)\times$$

$$\times\phi(x_{i+1})\cdots\phi(x_n)|0\rangle$$

Hence the Time ordered functions are Poincaré invariant if

$$\boxed{\begin{aligned} &1)\ \sum_{i=1}^{n} \langle 0|T\phi(x_1)\cdots(\partial_\mu\phi(x_i))\cdots\phi(x_n)|0\rangle = 0 \\ &2)\ \sum_{i=1}^{n} \langle 0|T\phi(x_1)\cdots((x_i^\mu\partial_i^\nu - x_i^\nu\partial_i^\mu - i\Sigma_i^{\mu\nu})\phi(x_i))\cdots \\ &\qquad\qquad \cdots\phi(x_n)|0\rangle = 0 \end{aligned}}$$

These are called the Poincaré Ward-Takahashi Identities (after Ward & Takahashi who first formulated symmetry (gauge) transformations in this manner in QED)

In short we can write this as

$$1)\ \sum_{i=1}^{n} \langle 0|T\phi(x_1)\cdots\delta_P^\mu\phi(x_i)\cdots\phi(x_n)|0\rangle = 0$$

$$2)\ \sum_{i=1}^{n} \langle 0|T\phi(x_1)\cdots\delta_M^{\mu\nu}\phi(x_i)\cdots\phi(x_n)|0\rangle = 0.$$

Of course we can transform such identities into our functional language by multiplying by sources J and summing,

$$0 = \sum_{n=1}^{\infty} \sum_{i=1}^{n} \int dx_1 \cdots dx_n \frac{i^n}{n!} J(x_1) \cdots J(x_n)$$

$$\langle 0 | T \phi(x_1) \cdots \delta\phi(x_i) \cdots \phi(x_n) | 0 \rangle$$

where δ is δ_P^μ or $\delta_M^{\mu\nu}$. Since x_i are dummy integration variables we can re-label n-times to obtain

$$= \sum_{n=1}^{\infty} \int dx\, J(x) \frac{i^n}{n!} n\, J(x_1) \cdots J(x_{n-1})\, dx_1 \cdots dx_{n-1}$$

$$\langle 0 | T\, \delta\phi(x)\, \phi(x_1) \cdots \phi(x_{n-1}) | 0 \rangle$$

$$= i \int dx\, J(x)\, \delta \frac{\delta}{i\delta J(x)} Z[J] = 0$$

Hence Poincaré invariance of the Green functions (and hence S-matrix) is expressed as a functional differential equation

1) Translation Invariance

$$\int dx\, J(x)\, \partial^\mu \frac{\delta}{\delta J(x)}\, Z[J] = 0$$

2) Lorentz Invariance

$$\int dx\, J(x) \left[(x^\mu \partial^\nu - x^\nu \partial^\mu) - i \Sigma_i^{\mu\nu} \right] \frac{\delta}{\delta J(x)}\, Z[J] = 0 .$$

These are the functional Ward Identities for Poincaré transformation. We can more simply express them by introducing the functional differential Ward Identity operators. we use the same symbol δ as the ordinary space-time differential generators without confusion (a post-script x on the space-time operator eliminates any confusion if necessary)

$$\delta \equiv \int dx\, J(x)\, \delta^x \frac{\delta}{\delta J(x)}$$

That is

$$\delta_P^\mu \equiv \int dx\, J(x)\, \delta_P^\mu \frac{\delta}{\delta J(x)} = \int dx\, J(x)\, \partial_x^\mu \frac{\delta}{\delta J(x)}$$

$$\delta_M^{\mu\nu} \equiv \int dx\, J(x)\, \delta_M^{\mu\nu} \frac{\delta}{\delta J(x)} = \int dx\, J(x)\, \times$$

$$\times \left[(x^\mu \partial^\nu - x^\nu \partial^\mu) - i \Sigma_i^{\mu\nu} \right] \frac{\delta}{\delta J(x)}$$

Since $\Lambda\,\delta^x$ represent the Poincaré algebra

$$[\delta_P^\mu, \delta_P^\nu] = 0$$

$$[\delta_M^{\mu\nu}, \delta_P^\lambda] = +(g^{\lambda\nu}\delta_P^\mu - g^{\mu\lambda}\delta_P^\nu)$$

$$[\delta_M^{\mu\nu}, \delta_M^{\rho\sigma}] = -[g^{\mu\rho}\delta_M^{\nu\sigma} + g^{\nu\sigma}\delta_M^{\mu\rho} - g^{\mu\sigma}\delta_M^{\nu\rho} - g^{\nu\rho}\delta_M^{\mu\sigma}]$$

(Recall that the Poincaré Algebra is represented by the Quantum generators $P^\mu, M^{\mu\nu}$

$$[P^\mu, P^\nu] = 0$$

$$[M^{\mu\nu}, P^\lambda] = i[g^{\lambda\nu}P^\mu - g^{\mu\lambda}P^\nu]$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i[g^{\mu\rho}M^{\nu\sigma} + g^{\nu\sigma}M^{\mu\rho} - g^{\mu\sigma}M^{\nu\rho} - g^{\nu\rho}M^{\mu\sigma}]$$

The Quantum generators are represented on the fields (and states) by space-time differential operators

$$[P^\mu, \phi(x)] = -i\delta_P^\mu \phi(x) = -i\partial^\mu \phi(x)$$

$$[M^{\mu\nu}, \phi(x)] = -i\,\delta_M^{\mu\nu}\,\phi(x) = -i\big[(x^\mu\partial^\nu - x^\nu\partial^\mu)$$
$$-i\,\Sigma^{\mu\nu}\big]\phi(x)$$

Since $P^\mu, M^{\mu\nu}$ obey the Poincaré algebra
So do δ_x's ; if $[Q_i, \phi(x)] = -i\,\delta_Q^i\,\phi(x)$
δ_Q^i = linear Diff. operator and $[Q_i, Q_j] = i f_{ijk} Q_k$

then
$$[[Q_i, Q_j], \phi(x)] = [Q_i, [Q_j, \phi(x)]]$$
$$- [Q_j, [Q_i, \phi(x)]]$$

$$= [Q_i, -i\delta_Q^j\phi(x)] - [Q_j, -i\delta_Q^i\phi(x)]$$

$$= -i\delta_Q^j\,[Q_i, \phi(x)] + i\delta_Q^i\,[Q_j, \phi(x)]$$

$$= -\delta_Q^j\delta_Q^i\,\phi(x) + \delta_Q^i\delta_Q^j\,\phi(x)$$

$$= [\delta_Q^i, \delta_Q^j]\,\phi(x)$$

but $[Q_i, Q_j] = i f_{ijk} Q_k$

So $\quad = i f_{ijk}[Q_k, \phi(x)] = f_{ijk}\,\delta_Q^k\,\phi(x)$

$$\Rightarrow \boxed{[\delta_Q^i, \delta_Q^j] = f_{ijk}\,\delta_Q^k}$$

Here the functional differential Ward Identity operators δ obey the Poincaré algebra also

i.e. $\delta^i_Q = \int dx\, J(x)\, \delta^{xi}_Q \frac{\delta}{\delta J(x)}$

$$[\delta^i_Q, \delta^j_Q] = \int dx\, J(x)\, \delta^{xi}_Q \frac{\delta}{\delta J(x)} \int dy\, J(y)\, \delta^{yj}_Q \frac{\delta}{\delta J(y)} - (i \rightleftarrows j)$$

$$= \int dx\, J(x)\, \delta^{xi}_Q \int dy\, \delta(x-y)\, \delta^{yj}_Q \frac{\delta}{\delta J(y)} - (i \rightleftarrows j)$$

$$= \int dx\, J(x) [\delta^{xi}_Q, \delta^{xj}_Q] \frac{\delta}{\delta J(x)}$$

$$= f_{ijk} \int dx\, J(x)\, \delta^{xk}_Q \frac{\delta}{\delta J(x)}$$

$$= f_{ijk}\, \delta^k_Q \quad \checkmark .$$

Then for the Poincaré algebra we have

$$[\delta^\mu_P, \delta^\nu_P] = 0$$

$$[\delta^{\mu\nu}_M, \delta^\lambda_P] = +(g^{\lambda\nu} \delta^\mu_P - g^{\lambda\mu} \delta^\nu_P)$$

$$[\delta^{\mu\nu}_M, \delta^{\rho\sigma}_M] = -(g^{\mu\rho} \delta^{\nu\sigma}_M + g^{\nu\sigma} \delta^{\mu\rho}_M$$

$$-g^{\mu\sigma}\delta_M^{\nu\rho} - g^{\nu\rho}\delta_M^{\mu\sigma}]$$

We have represented the Poincaré group on functionals as functional differential operators – the WI operators.

Hence to summarize, the Poincaré transformation are generated on the Green functions by the WI functional differential operators $\delta_P^\mu, \delta_M^{\mu\nu}$ acting on the Green function generating functional $Z[J]$.

If the Green function are Poincaré invariant, then they obey the WI functional differential equations

$$\delta_P^\mu Z[J] = 0$$

$$\delta_M^{\mu\nu} Z[J] = 0.$$

This implies the Poincaré invariance of the S-operator.